A one-level additive Schwarz preconditioner for a discontinuous Petrov-Galerkin method

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1 A discontinuous Petrov-Galerkin method for a model Poisson problem

Discontinuous Petrov-Galerkin (DPG) methods are new discontinuous Galerkin methods [3, 4, 5, 6, 7, 8] with interesting properties. In this article we consider a domain decomposition preconditioner for a DPG method for the Poisson problem.

Let Ω be a polyhedral domain in $\mathbb{R}^d$ $(d = 2,3)$, Ω_h be a simplicial triangulation of Ω. Following the notation in [8], the model Poisson problem (in an ultraweak formulation) is to find $\mathscr{U} \in U$ such that

$$b(\mathscr{U},\mathscr{V}) = l(\mathscr{V}) \qquad \forall\, \mathscr{V} \in V,$$

where $U = [L_2(\Omega)]^d \times L_2(\Omega) \times H_0^{\frac{1}{2}}(\partial\Omega_h) \times H^{-\frac{1}{2}}(\partial\Omega_h)$, $V = H(\mathrm{div};\Omega_h) \times H^1(\Omega_h)$,

$$\begin{aligned} b(\mathscr{U},\mathscr{V}) = \int_\Omega \boldsymbol{\sigma}\cdot\tau\,dx - \sum_{K\in\Omega_h}\int_K u\,\mathrm{div}\,\tau\,dx + \sum_{K\in\Omega_h}\int_{\partial K}\hat{u}\,\tau\cdot n\,ds \\ - \sum_{K\in\Omega_h}\int_K \boldsymbol{\sigma}\cdot\mathrm{grad}\,v\,dx + \sum_{K\in\Omega_h}\int_{\partial K} v\,\hat{\boldsymbol{\sigma}}_n\,ds \end{aligned}$$

for $\mathscr{U} = (\boldsymbol{\sigma}, u, \hat{u}, \hat{\boldsymbol{\sigma}}_n) \in U$ and $\mathscr{V} = (\tau, v) \in V$, and $l(\mathscr{V}) = \int_\Omega f v\,dx$.

Here $H_0^{1/2}(\partial\Omega_h)$ (resp. $H^{-1/2}(\partial\Omega_h)$) is the subspace of $\prod_{K\in\Omega_h} H^{1/2}(\partial K)$ (resp. $\prod_{K\in\Omega_h} H^{-1/2}(\partial K)$) consisting of the traces of functions in $H_0^1(\Omega)$ (resp. traces of the normal components of vector fields in $H(\mathrm{div};\Omega)$), and $H(\mathrm{div};\Omega_h)$ (resp. $H^1(\Omega_h)$) is the space of piecewise $H(\mathrm{div})$ vector fields (resp. H^1 functions). The inner product on V is given by

$$\big((\tau_1,v_1),(\tau_2,v_2)\big)_V = \sum_{K\in\Omega_h}\int_K [\tau_1\cdot\tau_2 + \mathrm{div}\,\tau_1\,\mathrm{div}\,\tau_2 + v_1 v_2 + \mathrm{grad}\,v_1\cdot\mathrm{grad}\,v_2]\,dx.$$

The DPG method for the Poisson problem computes $\mathscr{U}_h \in U_h$ such that

$$b(\mathscr{U}_h,\mathscr{V}) = l(\mathscr{V}) \qquad \forall\, \mathscr{V} \in V_h. \tag{1}$$

Here the trial space $U_h\,(\subset U)$ is defined by

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$$U_h = \prod_{K\in\Omega_h} [P_m(K)]^d \times \prod_{K\in\Omega_h} P_m(K) \times \tilde{P}_{m+1}(\partial\Omega_h) \times P_m(\partial\Omega_h),$$

$P_m(K)$ is the space of polynomials of total degree $\le m$ on an element K, $\tilde{P}_{m+1}(\partial\Omega_h) = H_0^{1/2}(\partial\Omega_h) \cap \prod_{K\in\Omega_h} \tilde{P}_{m+1}(\partial K)$, where $\tilde{P}_{m+1}(\partial K)$ is the restriction of $P_{m+1}(K)$ to ∂K, and $P_m(\partial\Omega_h) = H^{-1/2}(\partial\Omega_h) \cap \prod_{K\in\Omega_h} P_m(\partial K)$, where $P_m(\partial K)$ is the space of piecewise polynomials on the faces of K with total degree $\le m$.

Let $V^r = \{(\tau, v) \in V : \tau|_K \in [P_{m+2}(K)]^d, v|_K \in P_r(K)\ \forall K \in \Omega_h\}$ for some $r \ge m+d$. The discrete trial-to-test map $T_h : U_h \longrightarrow V^r$ is defined by

$$(T_h\mathscr{U}_h, \mathscr{V})_V = b(\mathscr{U}_h, \mathscr{V}), \quad \forall \mathscr{U}_h \in U_h,\ \mathscr{V} \in V^r,$$

and the test space V_h is T_hU_h.

We can rewrite (1) as $a_h(\mathscr{U}_h, \mathscr{W}) = l(T_h\mathscr{W})$ for all $\mathscr{W} \in U_h$, where

$$a_h(\mathscr{U}, \mathscr{W}) = b_h(\mathscr{U}, T_h\mathscr{W}) = (T_h\mathscr{U}, T_h\mathscr{W})_V$$

is an SPD bilinear form on $V_h \times V_h$, and we define an operator $A_h : U_h \longrightarrow U_h'$ by

$$\langle A_h\mathscr{U}, \mathscr{W}\rangle = a_h(\mathscr{U}, \mathscr{W}) \qquad \forall \mathscr{U}, \mathscr{W} \in U_h. \tag{2}$$

Our goal is to develop a one-level additive Schwarz preconditioner for A_h (cf. [9]).

To avoid the proliferation of constants, we will use the notation $A \lesssim B$ (or $B \gtrsim A$) to represent the inequality $A \le (\text{constant}) \times B$, where the positive constant only depends on the shape regularity of Ω_h and the polynomial degrees m and r. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

A fundamental result in [8] is the equivalence

$$a_h(\mathscr{U}, \mathscr{U}) \approx \|\sigma\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)} + \|\hat{u}\|^2_{H^{1/2}(\partial\Omega_h)} + \|\hat{\sigma}_n\|^2_{H^{-1/2}(\partial\Omega_h)} \tag{3}$$

that holds for all $\mathscr{U} = (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U_h$, where

$$\|\hat{u}\|^2_{H^{1/2}(\partial\Omega_h)} = \sum_{K\in\Omega_h} \|\hat{u}\|^2_{H^{1/2}(\partial K)} = \sum_{K\in\Omega_h} \inf_{w\in H^1(K),\, w|_{\partial K}=\hat{u}} \|w\|^2_{H^1(K)}, \tag{4}$$

$$\|\hat{\sigma}_n\|^2_{H^{-1/2}(\partial\Omega_h)} = \sum_{K\in\Omega_h} \|\hat{\sigma}_n\|^2_{H^{-1/2}(\partial K)} = \sum_{K\in\Omega_h} \inf_{q\in H(\mathrm{div};K),\, q\cdot n|_{\partial K}=\hat{\sigma}_n} \|q\|^2_{H(\mathrm{div};K)}. \tag{5}$$

Therefore the analysis of domain decomposition preconditioners for A_h requires a better understanding of the norms $\|\cdot\|_{H^{1/2}(\partial K)}$ and $\|\cdot\|_{H^{-1/2}(\partial K)}$ on the discrete spaces $\tilde{P}_{m+1}(\partial K)$ and $P_m(\partial K)$.

2 Explicit Expressions for the Norms on $\tilde{P}_{m+1}(\partial K)$ and $P_m(\partial K)$

Lemma 1. *We have*

$$\|\tilde{\zeta}\|^2_{H^{1/2}(\partial K)} \approx h_K\Big(\|\tilde{\zeta}\|^2_{L_2(\partial K)} + \sum_{F\in\Sigma_K} |\tilde{\zeta}|^2_{H^1(F)}\Big) \qquad \forall\,\tilde{\zeta} \in \tilde{P}_{m+1}(\partial K),$$

where h_K is the diameter of K and Σ_K is the set of the faces of K.

Proof. Let $\mathscr{N}(K)$ be the set of nodal points of the P_{m+1} Lagrange finite element associated with K and $\mathscr{N}(\partial K)$ be the set of points in $\mathscr{N}(K)$ that are on ∂K.

Given any $\tilde{\zeta} \in \tilde{P}_{m+1}(\partial K)$, we define $\tilde{\zeta}_* \in P_{m+1}(K)$ by

$$\tilde{\zeta}_*(p) = \begin{cases} \tilde{\zeta}(p) & \text{if } p \in \mathscr{N}(\partial K),\\ \tilde{\zeta}_{\partial K} & \text{if } p \in \mathscr{N}(K)\setminus\mathscr{N}(\partial K), \end{cases} \tag{6}$$

where $\tilde{\zeta}_{\partial K}$ is the mean value of $\tilde{\zeta}$ over ∂K. Since $\tilde{\zeta}_* = \tilde{\zeta}$ on ∂K, we have

$$\|\tilde{\zeta}\|_{H^{1/2}(\partial K)} = \inf_{w\in H^1(K),\, w|_{\partial K}=\tilde{\zeta}} \|w\|_{H^1(K)} \le \|\tilde{\zeta}_*\|_{H^1(K)}. \tag{7}$$

Suppose $w \in H^1(K)$ satisfies $w = \tilde{\zeta}$ on ∂K. It follows from (6) and the trace theorem with scaling that

$$\|\tilde{\zeta}_*\|^2_{L_2(K)} \lesssim h_K\|\tilde{\zeta}\|^2_{L_2(\partial K)} = h_K\|w\|^2_{L_2(\partial K)} \lesssim \|w\|^2_{H^1(K)}, \tag{8}$$

and, by standard estimates,

$$\begin{aligned} |\tilde{\zeta}_*|^2_{H^1(K)} = |\tilde{\zeta}_* - \tilde{\zeta}_{\partial K}|^2_{H^1(K)} &\lesssim h_K^{-1}\|\tilde{\zeta}_* - \tilde{\zeta}_{\partial K}\|^2_{L_2(\partial K)}\\ &= h_K^{-1}\|w - w_{\partial K}\|^2_{L_2(\partial K)} \lesssim |w|^2_{H^1(K)}. \end{aligned} \tag{9}$$

Combining (7)–(9), we have $\|\tilde{\zeta}\|^2_{H^{1/2}(\partial K)} \approx \|\tilde{\zeta}_*\|^2_{H^1(K)}$. The lemma then follows from (6), the equivalence of norms on finite dimensional spaces and scaling. □

Lemma 2. *We have*

$$\|\zeta\|^2_{H^{-1/2}(\partial K)} \approx h_K\|\zeta\|^2_{L_2(\partial K)} + h_K^{-d}\Big(\int_{\partial K}\zeta\,ds\Big)^2 \qquad \forall\,\zeta \in P_m(\partial K).$$

Proof. We begin with the reference simplex $\hat{K}$. Let $RT_m(\hat{K})$ be the m-th order Raviart-Thomas space (cf. [2]). Given any $\zeta \in P_m(\partial\hat{K})$, we introduce a (nonempty) subspace $RT_m(\hat{K},\zeta) = \{q \in RT_m(\hat{K}) :\ q\cdot n = \zeta \text{ on } \partial\hat{K} \text{ and } \operatorname{div} q \in P_0(\hat{K})\}$ of $RT_m(\hat{K})$.

Let $\zeta_* \in RT_m(\hat{K},\zeta)$ be defined by

$$\zeta_* = \min_{q\in RT_m(\hat{K},\zeta)} \|q\|_{L_2(\hat{K})}.$$

Then the map $\hat{S} : P_m(\partial\hat{K}) \longrightarrow RT_m(\hat{K})$ that maps ζ to ζ_* is linear and one-to-one, and we have $(\hat{S}\zeta)\cdot n = \zeta$ on $\partial\hat{K}$, $\operatorname{div}(\hat{S}\zeta) \in P_0(\hat{K})$ and

$$\|\hat{S}\zeta\|_{L_2(\hat{K})} \approx \|\zeta\|_{L_2(\partial\hat{K})} \qquad \forall\,\zeta \in P_m(\partial\hat{K}). \tag{10}$$

Let $\zeta_1,\ldots,\zeta_{N_m}$ be a basis of $P_m(\partial\hat{K})$ and $1=\phi_1,\ldots,\phi_{N_m}\in H^{1/2}(\partial\hat{K})$ satisfy $\det\left[\int_{\partial\hat{K}}\zeta_i\phi_j\,d\hat{s}\right]_{1\le i,j\le N_m}\neq 0$. We define the map $\hat{Q}:H(\mathrm{div};\hat{K})\longrightarrow P_m(\partial\hat{K})$ by

$$\int_{\partial\hat{K}}(\hat{Q}q)\phi_j\,d\hat{s}=\langle q\cdot n,\phi_j\rangle_{H^{-1/2}(\partial\hat{K})\times H^{1/2}(\partial\hat{K})}\qquad\text{for}\quad 1\le j\le N_m.$$

It follows from the definition of $\hat{Q}$ that $\|\hat{Q}q\|_{L_2(\partial\hat{K})}\lesssim\|q\|_{H(\mathrm{div};\hat{K})}$ for all $q\in H(\mathrm{div};\hat{K})$, and $\hat{Q}q=\zeta$ if $q\cdot n=\zeta\in P_m(\partial\hat{K})$, in which case

$$\|\hat{S}\zeta\|_{L_2(\hat{K})}\lesssim\|\zeta\|_{L_2(\partial\hat{K})}=\|\hat{Q}q\|_{L_2(\partial\hat{K})}\lesssim\|q\|_{H(\mathrm{div};\hat{K})}. \tag{11}$$

Moreover, since $\phi_1=1$, we have

$$\int_{\hat{K}}\mathrm{div}\,(\hat{S}\zeta)\,d\hat{x}=\int_{\partial\hat{K}}(\hat{Q}q)1d\hat{s}=\langle q\cdot n,1\rangle_{H^{-1/2}(\partial\hat{K})\times H^{1/2}(\partial\hat{K})}=\int_{\hat{K}}\mathrm{div}\,q\,d\hat{x}$$

and hence

$$\|\mathrm{div}\,(\hat{S}\zeta)\|_{L_2(\hat{K})}\lesssim\|\mathrm{div}\,q\|_{L_2(\hat{K})}. \tag{12}$$

Now we turn to a general simplex K. It follows from (10)–(12) and standard properties of the Piola transform for $H(\mathrm{div})$ (cf. [10]) that there exists a linear map $S:P_m(\partial K)\longrightarrow RT_m(K)$ with the following properties:
(i) $(S\zeta)\cdot n=\zeta$ and hence

$$\|\zeta\|_{H^{-1/2}(\partial K)}=\inf_{q\in H(\mathrm{div};K),\,q\cdot n|_{\partial K}=\zeta}\|q\|_{H(\mathrm{div};K)}\le\|S\zeta\|_{H(\mathrm{div};K)}\qquad\forall\,\zeta\in P_m(\partial K),$$

(ii) for any $q\in H(\mathrm{div};K)$ such that $q\cdot n=\zeta$, we have

$$\|S\zeta\|_{H(\mathrm{div};K)}\lesssim\|q\|_{H(\mathrm{div};K)},$$

(iii) $\mathrm{div}\,(S\zeta)\in P_0(K)$ and hence

$$\int_K\mathrm{div}\,(S\zeta)\,dx=\int_{\partial K}\zeta\,ds\quad\text{or}\quad\|\mathrm{div}\,(S\zeta)\|^2_{L_2(K)}=\Big(\int_{\partial K}\zeta\,ds\Big)^2/|K|,$$

(iv) we have

$$h_K^{-d}\|S\zeta\|^2_{L_2(K)}\approx h_K^{-(d-1)}\|\zeta\|^2_{L_2(\partial K)}.$$

Properties (i)–(iv) then imply

$$\|\zeta\|^2_{H^{-1/2}(\partial K)}\approx\|S\zeta\|^2_{H(\mathrm{div};K)}\approx h_K\|\zeta\|^2_{L_2(\partial K)}+h_K^{-d}\Big(\int_{\partial K}\zeta\,ds\Big)^2.\qquad\square$$

3 A Domain Decomposition Preconditioner

Let Ω be partitioned into overlapping subdomains $\Omega_1,\ldots,\Omega_J$ that are aligned with Ω_h. The overlap among the subdomains is measured by δ and we assume (cf. [11]) there is a partition of unity $\theta_1,\ldots,\theta_J \in C^\infty(\bar{\Omega})$ that satisfies the usual properties: $\theta_j \geq 0$, $\sum_{j=1}^J \theta_j = 1$ on $\bar{\Omega}$, $\theta_j = 0$ on $\Omega \setminus \Omega_j$, and

$$\|\nabla\theta_j\|_{L_\infty(\Omega)} \lesssim \delta^{-1} \qquad \forall\, 1 \leq j \leq J. \tag{13}$$

We take the subdomain space to be $U_j = \{\mathscr{v} \in U_h : \mathscr{v} = 0 \text{ on } \Omega \setminus \Omega_j\}$. Let $\mathscr{v} = (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U_h$. Then $\mathscr{v} \in U_j$ if and only if (i) σ and u vanish on every K outside Ω_j and (ii) $\hat{u}$ and $\hat{\sigma}_n$ vanish on ∂K for every K outside Ω_j. We define $a_j(\cdot,\cdot)$ to be the restriction of $a_h(\cdot,\cdot)$ on $U_j \times U_j$. Let $A_j : U_j \longrightarrow U_j'$ be defined by

$$\langle A_j \mathscr{v}_j, \mathscr{w}_j\rangle = a_j(\mathscr{v}_j, \mathscr{w}_j) \qquad \forall\, \mathscr{v}_j, \mathscr{w}_j \in U_j. \tag{14}$$

It follows from (3) that

$$a_j(\mathscr{v}_j,\mathscr{v}_j) \approx \|\sigma_j\|^2_{L_2(\Omega_j)} + \|u_j\|^2_{L_2(\Omega_j)} + \|\hat{u}_j\|^2_{H^{1/2}(\partial\Omega_{j,h})} + \|\hat{\sigma}_{n,j}\|^2_{H^{-1/2}(\partial\Omega_{j,h})}, \tag{15}$$

where $\mathscr{v}_j = (\sigma_j, u_j, \hat{u}_j, \hat{\sigma}_{n,j}) \in U_j$, $\Omega_{j,h}$ is the triangulation of Ω_j induced by Ω_h and the norms $\|\cdot\|_{H^{1/2}(\partial\Omega_{j,h})}$ and $\|\cdot\|_{H^{-1/2}(\partial\Omega_{j,h})}$ are analogous to those in (4) and (5).

Let $I_j : U_j \longrightarrow U_h$ be the natural injection. The one-level additive Schwarz preconditioner $B_h : U_h' \longrightarrow U_h$ is defined by

$$B_h = \sum_{j=1}^J I_j A_j^{-1} I_j^t.$$

Lemma 3. *We have*

$$\lambda_{\min}(B_h A_h) \gtrsim \delta^2.$$

Proof. Let $I_{h,1}$, $I_{h,2}$, $I_{h,3}$ and $I_{h,4}$ be the nodal interpolation operators for the components $\prod_{K\in\Omega_h}\left[P_m(K)\right]^d$, $\prod_{K\in\Omega_h} P_m(K)$, $\tilde{P}_{m+1}(\partial\Omega_h)$ and $P_m(\partial\Omega_h)$ of U_h respectively. Given any $\mathscr{v} = (\sigma, u, \hat{u}, \hat{\sigma}_n) \in U_h$, we define $\mathscr{v}_j \in U_j$ by

$$\mathscr{v}_j = \big(I_{h,1}(\theta_j\sigma), I_{h,2}(\theta_j u), I_{h,3}(\theta_j\hat{u}), I_{h,4}(\theta_j\hat{\sigma}_n)\big).$$

Then we have $\mathscr{v} = \sum_{j=1}^J \mathscr{v}_j$ and, in view of (14) and (15),

$$\begin{aligned}\langle A_j\mathscr{v}_j,\mathscr{v}_j\rangle \approx{}& \|I_{h,1}(\theta_j\sigma)\|^2_{L_2(\Omega_j)} + \|I_{h,2}(\theta_j u)\|^2_{L_2(\Omega_j)}\\ &+ \|I_{h,3}(\theta_j\hat{u})\|^2_{H^{1/2}(\partial\Omega_{j,h})} + \|I_{h,4}(\theta_j\hat{\sigma}_n)\|^2_{H^{-1/2}(\partial\Omega_{j,h})}.\end{aligned} \tag{16}$$

The following bounds for the first two terms on the right-hand side of (16) are straightforward:

$$\|I_{h,1}(\theta_j\sigma)\|^2_{L_2(\Omega_j)} \lesssim \|\sigma\|^2_{L_2(\Omega_j)} \quad \text{and} \quad \|I_{h,2}(\theta_j u)\|^2_{L_2(\Omega_j)} \lesssim \|u\|^2_{L_2(\Omega_j)}. \tag{17}$$

We will use Lemma 1 and Lemma 2 to derive the following bounds

$$\|I_{h,3}(\theta_j\hat{u})\|^2_{H^{1/2}(\partial\Omega_{j,h})} \lesssim \delta^{-2}\|\hat{u}\|^2_{H^{1/2}(\partial\Omega_{j,h})}, \tag{18}$$

$$\|I_{h,4}(\theta_j\hat{\sigma}_n)\|^2_{H^{-1/2}(\partial\Omega_{j,h})} \lesssim \delta^{-2}\|\hat{\sigma}_n\|^2_{H^{-1/2}(\partial\Omega_{j,h})}. \tag{19}$$

Let $K \in \Omega_{j,h}$. It follows from Lemma 1, (13) and standard discrete estimates that

$$\begin{aligned}
\|I_{h,3}(\theta_j\hat{u})\|^2_{H^{1/2}(\partial K)} &\approx h_K\Big(\|I_{h,3}(\theta_j\hat{u})\|^2_{L_2(\partial K)} + \sum_{F\in\Sigma_K}|I_{h,3}(\theta_j\hat{u})|^2_{H^1(F)}\Big)\\
&\lesssim h_K\|\hat{u}\|^2_{L_2(\partial K)} + h_K\sum_{F\in\Sigma_K}\big(\|\nabla\theta_j\|^2_{L_\infty(\Omega)}\|\hat{u}\|^2_{L_2(F)} + \|\theta_j\|^2_{L_\infty(\Omega)}|\hat{u}|^2_{H^1(F)}\big)\\
&\lesssim h_K\|\hat{u}\|^2_{L_2(\partial K)} + h_K\delta^{-2}\|\hat{u}\|^2_{L_2(\partial K)} + h_K\sum_{F\in\Sigma_K}|\hat{u}|^2_{H^1(F)} \lesssim \delta^{-2}\|\hat{u}\|^2_{H^{1/2}(\partial K)}.
\end{aligned}$$

Summing up this estimate over all the simplexes in $\Omega_{j,h}$ yields (18).

Similarly, it follows from Lemma 2 and (13) that

$$\begin{aligned}
\|I_{h,4}(\theta_j\hat{\sigma}_n)\|^2_{H^{-1/2}(\partial\hat{K})} &\approx h_K\|I_{h,4}(\theta_j\hat{\sigma}_n)\|^2_{L_2(\partial K)} + h_K^{-d}\Big(\int_{\partial K} I_{h,4}(\theta_j\hat{\sigma}_n)\,ds\Big)^2\\
&\lesssim h_K\|\hat{\sigma}_n\|^2_{L_2(\partial K)} + h_K^{-d}\Big(\int_{\partial K} I_{h,4}\big[(\theta_j-\theta_j^K)\hat{\sigma}_n\big]\,ds\Big)^2 + h_K^{-d}(\theta_j^K)^2\Big(\int_{\partial K}\hat{\sigma}_n\,ds\Big)^2\\
&\lesssim h_K\|\hat{\sigma}_n\|^2_{L_2(\partial K)} + h_K\delta^{-2}\|\hat{\sigma}_n\|^2_{L_2(\partial K)} + h_K^{-d}\Big(\int_{\partial K}\hat{\sigma}_n\,ds\Big)^2 \lesssim \delta^{-2}\|\hat{\sigma}_n\|^2_{H^{-1/2}(\partial K)},
\end{aligned}$$

where θ_j^K is the mean value of σ_j over K. Summing up this estimate over all the simplexes in $\Omega_{j,h}$ gives us (19).

Putting (2), (3) and (16)–(19) together we find $\sum_{j=1}^J\langle A_j\mathscr{U}_j,\mathscr{U}_j\rangle \lesssim \delta^{-2}\langle A_h\mathscr{U},\mathscr{U}\rangle$, which implies $\lambda_{\min}(B_hA_h) \gtrsim \delta^2$ by the standard theory of additive Schwarz preconditioners [11]. $\square$

Combining Lemma 3 with the standard estimate $\lambda_{\max}(B_hA_h) \lesssim 1$, we obtain the following theorem.

Theorem 1. *We have*

$$\kappa(B_hA_h) = \frac{\lambda_{\max}(B_hA_h)}{\lambda_{\min}(B_hA_h)} \le C\delta^{-2},$$

where the positive constant C depends only on the shape regularity of Ω_h and the polynomial degrees m and r.

Remark 1. Theorem 1 is also valid for DPG methods based on tensor product finite elements.

4 Numerical results

We solve the Poisson problem on the square $(0,1)^2$ with exact solution $u = \sin(\pi x_1)\sin(\pi x_2)$ and uniform square meshes. The trial space is based on Q_1 polynomials for σ and u, P_2 polynomials for $\hat{u}$, and P_1 polynomials for $\hat{\sigma}_n$. We use bicubic polynomials for the space V^r in the construction of the trial-to-test map T_h.

The number of conjugate gradient iterations required to reduce the residual by 10^{10} are given in Table 1 for four overlapping subdomains. The linear growth of the number of iterations for the unpreconditioned system is consistent with the condition number estimate $\kappa(A_h) \lesssim h^{-2}$ in [8]. Note that in this case the boundary of every subdomain has a nonempty intersection with $\partial\Omega$ and it is not difficult to use a discrete Poincaré inequality to show that the estimate in Theorem 1 can be improved to $\kappa(B_hA_h) \lesssim |\ln h|\delta^{-1}$. This is consistent with the observed growth of the number of iterations for the preconditioned system as δ decreases.

Table 1 Number of iterations for the Schwarz preconditioner with subdomain size $H = 1/2$.

h	δ	unpreconditioned	preconditioned
2^{-2}	2^{-2}	496	14
2^{-3}	2^{-3}	1556	17
	2^{-2}		14
2^{-4}	2^{-4}	3865	20
	2^{-3}		17
	2^{-2}		14
2^{-5}	2^{-5}	8793	27
	2^{-4}		20
	2^{-3}		18

In Table 2 we display the results for $h = 2^{-5}$ and various subdomain sizes H with $\delta = H/2$. The estimate $\kappa(B_hA_h) \lesssim \delta^{-2} \approx H^{-2}$ is consistent with the observed linear growth of the number of iterations for the preconditioned system as H decreases. Such a condition number estimate for the one-level additive Schwarz preconditioner is known to be sharp for standard finite element methods [1].

Table 2 Number of iterations with $h = 2^{-5}$ and various subdomain sizes H with $\delta = H/2$.

h	H	unpreconditioned	preconditioned
2^{-5}	2^{-1}	8793	15
	2^{-2}		25
	2^{-3}		45
	2^{-4}		89

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