Lower Dimensional Coarse Spaces for Domain Decomposition

Clark R. Dohrmann[1] and Olof B. Widlund[2]

1 Introduction

Coarse spaces are at the heart of many domain decomposition algorithms. Building on the foundation laid in [8], we have an ongoing interest in the development of coarse spaces based on energy minimization concepts [1]. Several different areas have been investigated recently, including compressible and almost compressible elasticity [3, 4], subdomains with irregular shapes [2, 11], problems in $H(curl)$ [6], and problems in $H(div)$ [12]. We also comment that there has been much recent complementary work to address problems having multiple materials in individual subdomains (see, e.g., [10, 9]).

The purpose of this study is to investigate a family of lower dimensional coarse spaces for scalar elliptic and elasticity problems. The basic idea involves the use of certain equivalence classes of nodes on subdomain boundaries. Coarse degrees of freedom are then associated with these classes, and the coarse basis functions are obtained from energy-minimizing extensions of subdomain boundary data into the subdomain interiors. We note in the context of a cube, domain decomposed into smaller cubical subdomains, that these classes are simply the subdomain vertices.

An analysis for scalar elliptic problems reveals that significant reductions in the coarse space dimension can often be achieved without sacrificing the favorable condition number estimates for larger coarse spaces. This can be important when the the memory and computational requirements associated with larger coarse spaces are prohibitive due to the use of large numbers of processors on a parallel computer. A multi-level approach could be used in such cases, but this may not always be possible or the best solution.

In the next section, we describe the nodal equivalence classes that are used in the construction of the coarse spaces. We then present algorithms for generating the coarse basis functions for different problem types in §3. An analysis for a scalar elliptic equation is provided in §4, and numerical examples are presented in §5.

[1] Computational Solid Mechanics and Structural Dynamics, Sandia National Laboratories, Albuquerque, New Mexico, 87185, USA. Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy's National Nuclear Security Administration under Contract DE-AC04-94-AL85000. e-mail: crdohrm@sandia.gov ·[2] Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012, USA. This work was supported in part by the National Science Foundation Grant DMS-1216564 and the U.S. Department of Energy under contracts DE-FG02-06ER25718. e-mail: widlund@cims.nyu.edu, http://cs.nyu.edu/cs/faculty/widlund/index.html

2 Coarse Nodes

Consider a domain Ω partitioned into non-overlapping subdomains $\Omega_1, \ldots, \Omega_N$. The set of all nodes common to two or more subdomains, excluding those with essential boundary conditions, is denoted by Γ_n. Let $\mathscr{S}_n$ denote the index set of subdomains containing node n. Two nodes $n_j, n_k \in \Gamma_n$ are related if $\mathscr{S}_{n_j} = \mathscr{S}_{n_k}$. As with FETI-DP or BDDC methods, we partition Γ_n into nodal equivalence classes based on this relation. Notice that for a decomposition of a cube into cubical subdomains that the nodal equivalence classes consist of faces (groups of nodes shared by the same two subdomains), edges (groups of nodes shared by the same four subdomains), and vertices (individual nodes shared by eight subdomains). For economy of words, we will henceforth use the abbreviation nec for nodal equivalence class.

Let $\mathscr{S}_{\mathscr{N}}$ denote the index set of subdomains for any node of nec $\mathscr{N}$. A nec $\mathscr{N}_j$ is said to be a child of nec $\mathscr{N}_k$ if $\mathscr{S}_{\mathscr{N}_j} \subset \mathscr{S}_{\mathscr{N}_k}$. Likewise, $\mathscr{N}_k$ is called a parent of $\mathscr{N}_j$ in this case. A nec is designated a coarse node if it is not the child of any other nec, and its coordinates are chosen as the centroid of its constituent nodes. Let $\mathscr{M}_i$ denote the set of all necs for Ω_i. Notice that each nec in $\mathscr{M}_i$ is either a coarse node or the child of at least one coarse node. Further, a coarse node c of Ω_i is also a coarse node of Ω_j for all $j \in \mathscr{S}_c$.

Notice that for the example decomposition described in the first paragraph of this section the coarse nodes are the subdomain vertices. If all necs are used as in [1], then there are approximately $(6/2 + 12/4 + 8/8)N = 7N$ necs associated with the coarse space. Likewise, if only subdomain edges and vertices are used as in [4], then there are approximately $(12/4 + 8/8)N = 4N$ necs. In contrast, the coarse space of this study is based on only about N coarse nodes.

3 Coarse Basis Functions

In this section, we describe how to construct coarse basis functions for scalar elliptic and elasticity problems in three dimensions. These coarse basis functions are fully continuous between adjacent subdomains, and we focus our attention on a single subdomain Ω_i. The support of a coarse basis function associated with coarse node c is the interior of the union of all $\bar{\Omega}_j$ with $j \in \mathscr{S}_c$.

The first step is to obtain a partition of unity for the nodes of $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$. Let $\mathscr{C}_{\mathscr{N}}$ denote the set of parent coarse nodes for nec $\mathscr{N}$. If $\mathscr{N}$ is itself a coarse node, then we take $\mathscr{C}_{\mathscr{N}} = \mathscr{N}$. For the simplest case, the partition of unity associated with node $n \in \mathscr{N}$ and coarse node $c \in \mathscr{C}_{\mathscr{N}}$ is chosen as

$$p_{nc} = 1/|\mathscr{C}_{\mathscr{N}}|. \tag{1}$$

One can easily confirm that $\sum_{c \in \mathscr{C}_{\mathscr{N}}} p_{nc} = 1$.

Notice from (1) that p_{nc} is the same for all $n \in \mathscr{N}$ and $c \in \mathscr{C}_{\mathscr{N}}$. This feature can cause abrupt changes in the coarse basis functions near nec boundaries, typically

resulting in a logarithmic factor $\log(H_i/h_i)$ in estimates for the energy of the coarse basis functions. Here, H_i is the diameter of Ω_i and h_i is the diameter of its smallest element.

In an attempt to avoid the logarithmic factor, we also consider a partition of unity originating from linear functions rather than constants. Define

$$a(n) := \left[\, 1 \; x_{n1} \; \cdots \; x_{nd} \,\right],$$

where x_{nj} is the j-coordinate of node n and d is the spatial dimension. Let the matrix $A_{\mathscr{N}}$ denote the row concatenation of $a(n)$ for all coarse nodes in $\mathscr{C}_{\mathscr{N}}$. Notice that the number of rows of $A_{\mathscr{N}}$ is the number of parent coarse nodes for $\mathscr{N}$ and that the number of columns is $d+1$. The origin is chosen as any one of the parent coarse nodes. With reference to (1), p_{nc} is now chosen as

$$p_{nc} = a(n) A_{\mathscr{N}}^{\dagger} e_c, \tag{2}$$

where $\dagger$ denotes the Moore-Penrose pseudo-inverse and e_c is a row vector with a single nonzero entry of 1 in the row of $A_{\mathscr{N}}$ corresponding to the coarse node c. As before, one can confirm that $\sum_{c \in \mathscr{C}_{\mathscr{N}}} p_{nc} = 1$. We note if $a(n)$ is replaced by only its first column, then (2) simplifies to (1).

The energy of Ω_i is defined as $E_i(u_i) := u_i^T A_i u_i$, where u_i is a vector of nodal degrees of freedom (dofs) for Ω_i and A_i is the stiffness matrix for Ω_i. Let R_{in} select the rows of u_i for the dofs of node $n \in \mathscr{N}$. That is, $R_{in} u_i$ is the vector of dofs for node n. Let $\mathscr{N}_{ic}$ denote the set of nodes on Γ_i which have c as a parent coarse node and define

$$\Psi_{ic} := \sum_{n \in \mathscr{N}_{ic}} p_{nc} R_{in}^T N_{nc},$$

where the matrix N_{nc} is specified later for different problem types.

Let $R_{i\Gamma}$ and R_{iI} select the rows of u_i for the nodal dofs on Γ_i and the interior of Ω_i, respectively, and define

$$A_{i\Gamma\Gamma} := R_{i\Gamma} A_i R_{i\Gamma}^T, \quad A_{iI\Gamma} := R_{iI} A_i R_{i\Gamma}^T, \quad A_{iII} := R_{iI} A_i R_{iI}^T, \quad \text{etc.}$$

The coarse basis function associated with the coarse node c is given by

$$\Phi_{ic} = \Psi_{ic} - R_{iI}^T A_{iII}^{-1} A_{iI\Gamma} (R_{i\Gamma} \Psi_{ic}).$$

We note that the first term on the right hand side of this expression is the boundary data for the coarse basis function, while the second term is its energy-minimizing extension into the interior of Ω_i.

For scalar elliptic equations like the Poisson equation, we choose

$$N_{nc} = \left[\, 1 \,\right].$$

Remark 1. The coarse space in [2] is obtained by choosing the subdomain vertices *and* edges as the coarse nodes, and using the partition of unity given in (1). Similarly,

the smaller coarse space of [5] is obtained by choosing only the subdomain vertices as the coarse nodes and using the partition of unity given in (2).

For elasticity problems, N_{nc} is chosen as

$$N_{nc} = \begin{bmatrix} 1 & 0 & 0 & 0 & x^c_{n3} & -x^c_{n2} \\ 0 & 1 & 0 & -x^c_{n3} & 0 & x^c_{n1} \\ 0 & 0 & 1 & x^c_{n2} & -x^c_{n1} & 0 \end{bmatrix},$$

where x^c_{nj} is the j-coordinate of node n with the origin at the coarse node c. The first three columns of N_{nc} correspond to rigid body translations, while the final three columns correspond to rigid body rotations about c. We note the expression for N_{nc} can be adapted easily to accommodate finite element models with shell elements simply by adding three more rows to N_{nc}.

4 Analysis

In this section, we develop estimates for the energy of a coarse interpolant of u_i for a scalar elliptic equation. The diffusion coefficient $\rho_i > 0$ is assumed constant in Ω_i (see §4.2 of [13] for additional details). We will use the symbol u_i for both a finite element function and its vector representation in terms of nodal values. Similarly, ϕ_{ic} is the finite element function counterpart of Φ_{ic}.

For simplicity, we assume shape regular tetrahedral subdomains. In this case, the coarse basis functions for Ω_i based on (2) are identical to those for the standard P_1 linear tetrahedral element on Γ_i. Consequently, the coarse basis functions are also identical to the standard shape functions throughout Ω_i since a linear function minimizes energy for boundary data given by a linear function. We have the standard estimate

$$E_i(\phi_{ic}) \leq CH_i\rho_i. \tag{3}$$

Let $\bar{u}_i$, $\bar{u}_{\mathscr{F}}$, $\bar{u}_{\mathscr{E}}$ denote the mean of a finite element function u over the subdomain Ω_i, a subdomain face $\mathscr{F}$, and a subdomain edge $\mathscr{E}$, respectively. For a face $\mathscr{F}$ of Ω_i, it follows from the a trace theorem and a Poincaré inequality that

$$\rho_i H_i |\bar{u}_{\mathscr{F}} - \bar{u}_i|^2 \leq CE_i(u_i). \tag{4}$$

Similarly, for an edge $\mathscr{E}$ of Ω_i, we find using a discrete Sobolev inequality (see, e.g., Lemma 4.16 of [13]) that

$$\rho_i H_i |\bar{u}_{\mathscr{E}} - \bar{u}_i|^2 \leq C(1+\log(H_i/h_i))E_i(u_i). \tag{5}$$

Assumption 1: Let c be any vertex of Ω_i and $\mathscr{S}_c$ the index set of all subdomains containing c. Pick $j_c \in \mathscr{S}_c$ such that $\rho_{j_c} \geq \rho_j$ for all $j \in \mathscr{S}_c$. There exists a sequence $\{i = j^0_c, j^1_c, ..., j^p_c = j_c\}$ such that $\rho_i \leq C\rho_{j^\ell_c}$ and $\Omega_{j^{\ell-1}_c}$ and $\Omega_{j^\ell_c}$ have a face in common for all $\ell = 1, \ldots, p$ and $i = 1, \ldots, N$.

In other words, Assumption 1 means there is a face connected path between Ω_i and Ω_{j_c} such that the diffusion coefficient ρ_i is no greater than a constant times the diffusion coefficient of any subdomain along the path. This assumption appears to be essentially the same as the quasi-monotone assumption in [7].

Assumption 2: Using the same notation as in Assumption 1, there exists a sequence $\{i = j_c^0, j_c^1, ..., j_c^p = j_c\}$ such that $\rho_i \leq C\rho_{j_c^\ell}$ and $\Omega_{j_c^{\ell-1}}$ and $\Omega_{j_c^\ell}$ have an edge in common for all $\ell = 1, \ldots, p$ and $i = 1, \ldots, N$.

Notice that Assumption 2 is weaker than Assumption 1 since we have more options to continue at every step in the construction of a path. Our coarse interpolant of u_i for Ω_i is chosen as

$$u_{ic} = \sum_{c \in \mathscr{M}_{ic}} \bar{u}_{j_c} \phi_{ic}, \tag{6}$$

where $\mathscr{M}_{ic}$ is the set of subdomain vertices for Ω_i. Let $\mathscr{F}_{ij}$ denote the face common to Ω_i and Ω_j. Since the coarse basis functions for Ω_i can approximate constants exactly on Γ_i and also minimize the energy, it follows from a Poincaré inequality that

$$E_i(\sum_{c \in \mathscr{M}_{ic}} \bar{u}_i \phi_{ic}) \leq C E_i(u_i). \tag{7}$$

We next establish bounds for $E_i(u_{ic})$. Starting with

$$\bar{u}_i - \bar{u}_{j_c} = (\bar{u}_i - \bar{u}_{\mathscr{F}_{j_c^0 j_c^1}}) + \sum_{\ell=1}^{p-1} (\bar{u}_{\mathscr{F}_{j_c^{\ell-1} j_c^\ell}} - \bar{u}_{\mathscr{F}_{j_c^\ell j_c^{\ell+1}}}) + (\bar{u}_{\mathscr{F}_{j_c^{p-1} j_c^p}} - \bar{u}_{j_c}),$$

rewriting the term in the summation as

$$\bar{u}_{\mathscr{F}_{j_c^{\ell-1} j_c^\ell}} - \bar{u}_{\mathscr{F}_{j_c^\ell j_c^{\ell+1}}} = (\bar{u}_{\mathscr{F}_{j_c^\ell j_c^{\ell-1}}} - \bar{u}_{j_c^\ell}) - (\bar{u}_{\mathscr{F}_{j_c^\ell j_c^{\ell+1}}} - \bar{u}_{j_c^\ell}),$$

and using Assumption 1 and (4), we find

$$\rho_i H_i |\bar{u}_i - \bar{u}_{j_c}|^2 \leq C \sum_{j \in \mathscr{S}_c} E_j(u_j).$$

It then follows from (3) that

$$E_i((\bar{u}_i - \bar{u}_{j_c})\phi_{ic}) \leq C \sum_{j \in \mathscr{S}_c} E_j(u_j).$$

Finally, from (6), (7), and the triangle inequality, we obtain

$$E_i(u_{ic}) \leq C \sum_{j \in \mathscr{M}_i} E_j(u_j),$$

where $\mathscr{M}_i$ is the index set of all subdomains adjacent to Ω_i. Summing contributions from all subdomains and noting that $|\mathscr{M}_i| < C$, we see that the energy of our coarse interpolant is uniformly bounded by the energy of u. That is, under Assumption 1,

$$\sum_{i=1}^{N} E_i(u_{ic}) \leq C \sum_{i=1}^{N} E_i(u_i). \tag{8}$$

By using (5) instead of (4) in the previous development, we find under the less restrictive Assumption 2 that

$$\sum_{i=1}^{N} E_i(u_{ic}) \leq C(1+\log(H/h)) \sum_{i=1}^{N} E_i(u_i), \tag{9}$$

where $H/h := \max_i(H_i/h_i)$.

If the coarse basis functions originate from (1) rather than (2), then it follows from elementary estimates and Lemma 4.25 of [13] that an additional factor of $1+\log(H_i/h_i)$ will appear on the right-hand-side of (3). Thus, this additional factor will also be present in (8) and (9). The same also holds for hexahedral subdomains even when (2) is used since a linear function cannot interpolate a function at all four nodes of a quadrilateral planar face.

With the estimates for our coarse interpolants in hand, we may now perform a local analysis for an overlapping additive Schwarz algorithm using basically the same approach as in [2] or [5]. This involves a partition of unity $\{\vartheta_i\}_{i=1}^N$ with $0 \leq \vartheta_i \leq 1$, $|\nabla \vartheta_i| \leq C/\delta_i$, and ϑ_i supported in the closure of the overlapping subdomain Ω_i'. Here, δ_i is the thickness of the part of Ω_i' which is common to its neighbors. Given an estimate of the form

$$\sum_{i=1}^{N} E_i(u_{ic}) \leq C f(H/h) \sum_{i=1}^{N} E_i(u_i),$$

the resulting condition number estimate for the preconditioned operator is given by

$$\kappa(M^{-1}A) \leq C f(H/h)(1+H/\delta), \tag{10}$$

where $H/\delta := \max_i H_i/\delta_i$. Comparing (10) with (8) and (9), we that $f(H/h)$ is 1 and $1+\log(H/h)$ under Assumptions 1 and 2, respectively.

5 Numerical Examples

We consider a unit cube domain decomposed into either smaller cubical subdomains or irregular-shaped subdomains obtained from a mesh partitioner for a scalar elliptic equation; an analysis and results for elasticity will appear in a forthcoming study. The numbers of iterations and condition number estimates from the conjugate gradient algorithm appear under the headings *iter* and *cond* in the tables. All results are for homogeneous essential boundary conditions on one face of the cube, a random right-hand-side vector, and a relative residual solver tolerance of 10^{-8}.

The results in Table 1 are for 64 cubical subdomains and a fixed dimensionless overlap H/δ. By plotting condition numbers versus $\log(H/h)$, it appears that the

line segment slopes are bounded above by constants as H/h increases for both the constant and checkerboard material properties. Moreover, these line segment slopes for constant material properties and the linear partition of unity in (2) appear to decrease with increasing H/h, while those for (1) appear to approach a constant value. These observations are consistent with the analysis. We note for a vertex coarse space, as used in this example, a much less favorable condition number estimate of $C(H/h)(1+\log(H/h))^2$ holds for FETI-DP and BDDC algorithms (cf. Algorithm A in §6.4.2 of [13]).

Table 1 Results for constant and checkerboard arrangements of subdomain material properties ($\rho_i = 1$ or $\rho_i = 10^4$) for partitions of unity based on (1) and (2). The overlap $H/\delta \approx 4$ is held fixed while H/h varies.

	constant				checkerboard			
	p_{nc} (1)		p_{nc} (2)		p_{nc} (1)		p_{nc} (2)	
H/h	iter	cond	iter	cond	iter	cond	iter	cond
8	40	29.0	37	25.2	37	39.9	35	29.7
12	43	33.3	38	27.7	40	46.4	37	32.5
16	45	36.4	39	29.3	40	50.9	38	34.4
20	45	38.8	39	30.5	41	54.1	38	35.7

For the final example, we consider a mesh of 48^3 elements decomposed into different numbers of subdomains using a mesh partitioner. Results in Table 2 show that the present coarse space dimensions are significantly smaller than those for the richer coarse space in [1]. Smaller dimensional coarse spaces result in reduced computational requirements for the coarse problem, and extend the range of problem sizes that can be solved effectively using a two-level method.

Table 2 Results for constant coefficients and a mesh with 48^3 elements decomposed using a mesh partitioner. The coarse space dimension is denoted by n_c and the overlap is for two layers of additional elements. The final row in the table is for a regular mesh decomposition into 64 identical subdomains.

	Ref. [1]			p_{nc} (1)			p_{nc} (2)		
N	n_c	iter	cond	n_c	iter	cond	n_c	iter	cond
63	831	45	21.3	166	46	22.5	166	40	15.7
64	863	45	21.5	174	46	22.5	174	41	16.4
65	916	46	21.1	189	46	21.7	189	40	16.6
$64^\star$	279	40	24.9	27	43	33.3	27	38	27.7

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