Surrogate Functional Based Subspace Correction Methods for Image Processing

Michael Hintermüller[1] and Andreas Langer[2]

1 Introduction

Recently in [4, 5, 6] subspace correction methods for non-smooth and non-additive problems have been introduced in the context of image processing, where the non-smooth and non-additive total variation (TV) plays a fundamental role as a regularization technique, since it preserves edges and discontinuities in images. We recall, that for $u \in L^1(\Omega)$, $V(u,\Omega) := \sup\left\{\int_\Omega u \operatorname{div}\phi \, dx : \phi \in [C_c^1(\Omega)]^2, \|\phi\|_\infty \leq 1\right\}$ is the variation of u. In the event that $V(u,\Omega) < \infty$ we denote $|Du|(\Omega) = V(u,\Omega)$ and call it the total variation of u in Ω [1].

In this paper, as in [6], we consider functionals, which consist of a non-smooth and non-additive regularization term and a weighted combination of an ℓ^1-term and a quadratic ℓ^2-term; see (1) below. This type of functional has been shown to be particularly efficient to eliminate simultaneously Gaussian and salt-and-pepper noise. In [6] an estimate of the distance of the limit point obtained from the proposed subspace correction method to the global minimizer is established. In that paper the exact subspace minimization problems are minimized, which are in general not easily solved. Therefore, in the present paper we analyse a subspace correction approach in which the subproblems are approximated by so-called *surrogate* functionals, as in [4, 5]. In this situation, as in [6], we are able to achieve an estimate for the distance of the computed solution to the real global minimizer. With the help of this estimate we show in our numerical experiments that the proposed algorithm generates a sequence which converges to the expected minimizer.

2 Notations

For the sake of brevity we consider a two dimensional setting only. We define $\Omega = \{x_1 < \ldots < x_N\} \times \{y_1 < \ldots < y_N\} \subset \mathbb{R}^2$, and $H = \mathbb{R}^{N\times N}$, where $N \in \mathbb{N}$. For $u \in H$ we write $u = u(x) = u(x_i, y_j)$, where $i, j \in \{1,\ldots,N\}$ and $x \in \Omega$. Let $h = x_{i+1} - x_i = y_{j+1} - y_j$ be the equidistant step-size. We define the scalar product of $u, v \in H$ by $\langle u,v\rangle_H = h^2 \sum_{x\in\Omega} u(x)v(x)$ and the scalar product of $p, q \in H^2$ by $\langle p,q\rangle_{H^2} = h^2 \sum_{x\in\Omega} \langle p(x), q(x)\rangle_{\mathbb{R}^2}$ with $\langle z,w\rangle_{\mathbb{R}^2} = \sum_{j=1}^2 z_j w_j$ for every $z = (z_1,z_2) \in \mathbb{R}^2$ and $w =$

[1] Department of Mathematics, Humboldt-University of Berlin, Unter den Linden 6, 10099 Berlin, Germany, e-mail: hint@math.hu-berlin.de · [2] Institute for Mathematics and Scientific Computing, University of Graz, Heinrichstraße 36, A-8010 Graz, Austria, e-mail: andreas.langer@uni-graz.at

$(w_1,w_2)\in\mathbb{R}^2$. We also use $\|u\|_{\ell^p(\Omega)} = \left(h^2\sum_{x\in\Omega}|u(x)|^p\right)^{1/p}$, $1\le p<\infty$, $\|u\|_{\ell^\infty(\Omega)} = \sup_{x\in\Omega}|u(x)|$ and $\|\cdot\|$, when any norm can be taken.

The discrete gradient ∇u is denoted by $(\nabla u)(x) = ((\nabla u)^1(x),(\nabla u)^2(x))$ with $(\nabla u)^1(x) = \frac{1}{h}(u(x_{i+1},y_j)-u(x_i,y_j))$ if $i<N$ and $(\nabla u)^1(x)=0$ if $i=N$, and $(\nabla u)^2(x) = \frac{1}{h}(u(x_i,y_{j+1})-u(x_i,y_j))$ if $j<N$ and $(\nabla u)^2(x)=0$ if $j=N$, for all $x\in\Omega$. For $\omega\in H^2$ we define $\varphi:\mathbb{R}\to\mathbb{R}$ by $\varphi(|\omega|)(\Omega) := h^2\sum_{x\in\Omega}\varphi(|\omega(x)|)$, where $|z| = \sqrt{z_1^2+z_2^2}$. In particular we define the *total variation* of u by setting $\varphi(t)=t$ and $\omega=\nabla u$, i.e., $|\nabla u|(\Omega) := h^2\sum_{x\in\Omega}|\nabla u(x)|$.

For an operator T we denote by T^* its adjoint. Further we introduce the *discrete divergence* $\operatorname{div}: H^2\to H$ defined by $\operatorname{div} = -\nabla^*$ (∇^* is the adjoint of the gradient ∇), in analogy to the continuous setting. The symbol 1 indicates the constant vector with entry values 1 and 1_D is the characteristic function of $D\subset\Omega$.

For a convex functional $J: H\to\overline{\mathbb{R}}$, we define the *subdifferential* of J at $v\in H$ as the set valued mapping $\partial J(v) := \emptyset$ if $J(v)=\infty$ and $\partial J(v) := \{v^*\in H : \langle v^*, u-v\rangle_H + J(v)\le J(u) \quad \forall u\in H\}$ otherwise. It is clear from this definition that $0\in\partial J(v)$ if and only if v is a minimizer of J. Whenever the underlying space is important, then we write $\partial_{V_i}J$ or $\partial_H J$.

3 Subspace Correction Approaches

As in [6] we are interested in minimizing by means of subspace correction the following functional

$$J(u) = \alpha_S\|Su-g_S\|_{\ell^1(\Omega)} + \alpha_T\|Tu-g_T\|^2_{\ell^2(\Omega)} + \varphi(|\nabla u|)(\Omega), \tag{1}$$

where $S,T: H\to H$ are bounded linear operators, $g_S, g_T\in H$ are given data, and $\alpha_S,\alpha_T\ge 0$ with $\alpha_S+\alpha_T\ge\tau>0$. We assume that J is bounded from below and coercive, i.e., $\{u\in H : J(u)\le C\}$ is bounded in H for all constants $C>0$, in order to guarantee that (1) has minimizers. Moreover we assume that $\varphi:\mathbb{R}\to\mathbb{R}$ is a convex function, nondecreasing in $\mathbb{R}^+$ with (i) $\varphi(0)=0$ and (ii) $cz-b\le\varphi(z)\le cz+b$, for all $z\in\mathbb{R}^+$ for some constant $c>0$ and $b\ge 0$.

Note that for the particular example $\varphi(t)=t$, the third term in (1) becomes the well-known total variation of u in Ω and we call (1) the L^1-L^2-TV model.

Now we seek to minimize (1) by decomposing H into two subspaces V_1 and V_2 such that $H=V_1+V_2$. Note that a generalization to multiple splittings can be performed straightforwardly. However, here we will restrict ourselves to a decomposition into two domains only for simplicity. By V_i^c we denote the orthogonal complement of V_i in H and we define by $\pi_{V_i^c}$ the corresponding orthogonal projection onto V_i^c for $i=1,2$.

With this splitting we want to minimize J by suitable instances of the following alternating algorithm:

Choose an initial $u^{(0)} =: u_1^{(0)} + u_2^{(0)} \in V_1+V_2$, for example, $u^{(0)}=0$, and iterate

$$\begin{cases} u_1^{(n+1)} = \arg\min_{u_1 \in V_1} J(u_1 + u_2^{(n)}), \\ u_2^{(n+1)} = \arg\min_{u_2 \in V_2} J(u_1^{(n+1)} + u_2), \\ u^{(n+1)} := u_1^{(n+1)} + u_2^{(n+1)}. \end{cases} \tag{2}$$

Differently from the case in [6], where the authors solved the exact subspace minimization problems in (2), we suggest now to approximate the subdomain problems by so-called surrogate functionals (cf. [2, 3, 4, 5, 8]): Assume $a, u_i \in V_i$, $u_{-i} \in V_{-i}$, and define

$$\begin{aligned} J^s(u_i + u_{-i}, a + u_{-i}) := {} & J(u_i + u_{-i}) + \alpha_T \big(\delta \|u_i + u_{-i} - (a + u_{-i})\|^2_{\ell^2(\Omega)} \\ & - \|T(u_i + u_{-i} - (a + u_{-i}))\|^2_{\ell^2(\Omega)}\big) \\ = {} & J(u_i + u_{-i}) + \alpha_T \left(\delta \|u_i - a\|^2_{\ell^2(\Omega)} - \|T(u_i - a)\|^2_{\ell^2(\Omega)}\right) \end{aligned} \tag{3}$$

for $i = 1,2$ and $-i \in \{1,2\} \setminus \{i\}$, where $\delta > \|T\|^2$. Then an approximate solution to $\min_{u_i \in V_i} J(u_1 + u_2)$ is realized by using the following algorithm: For $u_i^{(0)} \in V_i$,

$$u_i^{(\ell+1)} = \arg\min_{u_i \in V_i} J^s(u_i + u_{-i}, u_i^{(\ell)} + u_{-i}), \quad \ell \geq 0,$$

where $u_{-i} \in V_{-i}$ for $i = 1,2$ and $-i \in \{1,2\} \setminus \{i\}$.

The alternating domain decomposition algorithm reads then as follows:

Choose an initial $u^{(0)} =: \tilde{u}_1^{(0)} + \tilde{u}_2^{(0)} \in V_1 + V_2$, for example, $u^{(0)} = 0$, and iterate

$$\begin{cases} \begin{cases} u_1^{(n+1,0)} = \tilde{u}_1^{(n)}, \\ u_1^{(n+1,\ell+1)} = \arg\min_{u_1 \in V_1} J^s(u_1 + \tilde{u}_2^{(n)}, u_1^{(n+1,\ell)} + \tilde{u}_2^{(n)}),\ \ell = 0, \ldots, L-1, \end{cases} \\ \begin{cases} u_2^{(n+1,0)} = \tilde{u}_2^{(n)}, \\ u_2^{(n+1,m+1)} = \arg\min_{u_2 \in V_2} J^s(u_1^{(n+1,L)} + u_2, u_2^{(n+1,m)} + u_1^{(n+1,L)}),\ m = 0, \ldots, M-1, \end{cases} \\ u^{(n+1)} := u_1^{(n+1,L)} + u_2^{(n+1,M)},\ \tilde{u}_1^{(n+1)} = \chi_1 \cdot u^{(n+1)},\ \tilde{u}_2^{(n+1)} = \chi_2 \cdot u^{(n+1)}, \end{cases} \tag{4}$$

where $\chi_1, \chi_2 \in H$ have the properties (i) $\chi_1 + \chi_2 = 1$ and (ii) $\chi_i \in V_i$ for $i = 1,2$. Let $\kappa := \max\{\|\chi_1\|_\infty, \|\chi_2\|_\infty\} < \infty$.

The parallel version of the algorithm in (4) reads as follows:

Choose an initial $u^{(0)} =: \tilde{u}_1^{(0)} + \tilde{u}_2^{(0)} \in V_1 + V_2$, for example, $u^{(0)} = 0$, and iterate

$$\begin{cases} \begin{cases} u_1^{(n+1,0)} = \tilde{u}_1^{(n)}, \\ u_1^{(n+1,\ell+1)} = \arg\min_{u_1 \in V_1} J^s(u_1 + \tilde{u}_2^{(n)}, u_1^{(n+1,\ell)} + \tilde{u}_2^{(n)}),\ \ell = 0,\dots,L-1, \end{cases} \\ \begin{cases} u_2^{(n+1,0)} = \tilde{u}_2^{(n)}, \\ u_2^{(n+1,m+1)} = \arg\min_{u_2 \in V_2} J^s(\tilde{u}_1^{(n)} + u_2, u_2^{(n+1,m)} + \tilde{u}_1^{(n)}),\ m = 0,\dots,M-1, \end{cases} \\ u^{(n+1)} := \frac{u_1^{(n+1,L)} + u_2^{(n+1,M)} + u^{(n)}}{2},\ \tilde{u}_1^{(n+1)} = \chi_1 \cdot u^{(n+1)},\ \tilde{u}_2^{(n+1)} = \chi_2 \cdot u^{(n+1)}. \end{cases} \tag{5}$$

Note that we prescribe a finite number L and M of inner iterations for each subspace, respectively. Hence we do not get a minimizer of the original subspace minimization problems in (2), but approximations of such minimizers. Moreover, observe that $u^{(n+1)} = \tilde{u}_1^{(n+1)} + \tilde{u}_2^{(n+1)}$, with $u_i^{(n+1,L)} \neq \tilde{u}_i^{(n+1)}$, for $i = 1,2$, in general.

We have that $u_1^{(n+1,L)} \in \arg\min_{u \in H} \left\{ J^s(u + \tilde{u}_2^{(n)}, u_1^{(n+1,L-1)} + \tilde{u}_2^{(n)}) : \pi_{V_1^c} u = 0 \right\}$. Then, by [7, Theorem 2.1.4, p. 305] there exists an $\eta_1^{(n+1)} \in \operatorname{Range}(\pi_{V_1^c})^* \simeq V_1^c$ such that

$$0 \in \partial_H J^s(\cdot + \tilde{u}_2^{(n)}, u_1^{(n+1,L-1)} + \tilde{u}_2^{(n)})(u_1^{(n+1,L)}) + \eta_1^{(n+1)}. \tag{6}$$

Analogously, we have that if $u_2^{(n+1,M)}$ is a minimizer of the second optimization problem in (4) or (5), then there exists an $\eta_2^{(n+1)} \in \operatorname{Range}(\pi_{V_2^c})^* \simeq V_2^c$ such that

$$0 \in \partial_H J^s(u_1^{(n+1,L)} + \cdot, u_1^{(n+1,L)} + \tilde{u}_2^{(n+1,M-1)})(u_2^{(n+1,M)}) + \eta_2^{(n+1)}, \text{ or} \tag{7}$$

$$0 \in \partial_H J^s(\tilde{u}_1^{(n,L)} + \cdot, \tilde{u}_1^{(n,L)} + \tilde{u}_2^{(n+1,M-1)})(u_2^{(n+1,M)}) + \eta_2^{(n+1)}, \tag{8}$$

respectively.

3.1 Convergence Properties

In this section we state convergence properties of the subspace correction methods in (4) and (5). In particular, the following three propositions are direct consequences of statements in [4, 5, 6].

Proposition 1. *The algorithms in (4) and (5) produce a sequence $(u^{(n)})_n$ in H with the following properties:*

1. $J(u^{(n)}) > J(u^{(n+1)})$ *for all $n \in \mathbb{N}$ (unless $u^{(n)} = u^{(n+1)}$);*
2. $\lim_{n\to\infty} \|u_1^{(n+1,\ell+1)} - u_1^{(n+1,\ell)}\|_{\ell^2(\Omega)} = 0$ *and* $\lim_{n\to\infty} \|u_2^{(n+1,m+1)} - u_2^{(n+1,m)}\|_{\ell^2(\Omega)} = 0$ *for all $\ell \in \{0,\dots,L-1\}$ and $m \in \{0,\dots,M-1\}$;*
3. $\lim_{n\to\infty} \|u^{(n+1)} - u^{(n)}\|_{\ell^2(\Omega)} = 0$*;*
4. *the sequence $(u^{(n)})_n$ has subsequences that converge in H.*

The proof of this proposition is analogous to the one in [5, Theorem 5.1].

Proposition 2. *The sequences $(\tilde{u}_i^{(n)})_n$ for $i = 1,2$ generated by the algorithm in (4) or (5) are bounded in H and hence have accumulation points $\tilde{u}_i^{(\infty)}$, respectively.*

Proof. By the boundedness of the sequence $(u^{(n)})_n$ we obtain $\|\tilde{u}_i^{(n)}\| = \|\chi_i u^{(n)}\| \leq \kappa \|u^{(n)}\| \leq C < \infty$ and hence $(\tilde{u}_i^{(n)})_n$ is bounded for $i = 1,2$. $\square$

Remark 1. From the previous proposition it directly follows by the coercivity assumption on J that the sequences $(u_1^{(n,\ell)})_n$ and $(u_2^{(n,m)})_n$ are bounded for all $\ell \in \{0,\dots,L\}$ and $m \in \{0,\dots,M\}$.

Proposition 3. *Let $u_1^{(\infty)}$, $u_2^{(\infty)}$, and $\tilde{u}_i^{(\infty)}$ be accumulation points of the sequences $(u_1^{(n,L)})_n$, $(u_2^{(n,M)})_n$, and $(\tilde{u}_i^{(n)})_n$ generated by the algorithms in (4) and (5), then $u_i^{(\infty)} = \tilde{u}_i^{(\infty)}$, for $i = 1,2$.*

One shows this statement analogous to the first part of the proof of [4, Theorem 5.7].

Moreover, as in [6] we are able to establish an estimate of the distance of the limit point obtained from the subspace correction method to the true global minimizer.

Theorem 1. *Let $\alpha_S \geq \tau$, u^* a minimizer of J, and $u^{(\infty)}$ an accumulation point of the sequence $(u^{(n)})_n$ generated by the algorithm in (4) or (5). Then we have that*

1. *$u^{(\infty)}$ is a minimizer of J or*
2. *there exists a constant $\beta > 0$ (independent of α_T) such that $\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)} \leq \beta$ or*
3. *if $\alpha_T < \frac{\gamma}{\beta^2\delta}$ for $0 < \gamma \leq J(u^{(\infty)}) - J(u^*)$, then $\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)} \leq \frac{\beta^2\|\hat{\eta}\|_{\ell^2(\Omega)}}{\gamma - \alpha_T\delta\beta^2}$, where $\|\hat{\eta}\|_{\ell^2(\Omega)} = \min\{\|\eta_1^{(\infty)}\|_{\ell^2(\Omega)}, \|\eta_2^{(\infty)}\|_{\ell^2(\Omega)}\}$ and $\eta_i^{(\infty)}$ is an accumulation point of the sequence $(\eta_i^{(n)})_n$ for $i = 1,2$ defined as in (6)-(8) respectively, or*
4. *if T^*T is positive definite with smallest Eigenvalue $\sigma > 0$, $\alpha_T > 0$ and $\|T\|^2 < \delta < 2\sigma$, then we have $\|u^* - u^{(\infty)}\|_{\ell^2(\Omega)} \leq \frac{\|\hat{\eta}\|_{\ell^2(\Omega)}}{\alpha_T(2\sigma-\delta)}$.*

Proof. Since $(u_1^{(n+1,L)})_n$, $(u_1^{(n+1,L-1)})_n$, and $(\tilde{u}_2^{(n)})_n$ are bounded and based on the fact that $\partial J^s(\xi,\tilde{\xi})$ is compact for any $\xi,\tilde{\xi} \in H$ we obtain that $(\eta_1^{(n)})_n$ is bounded, cf. [6, Corollary 4.7]. By noting that $(u_1^{(n+1,L)})_n$ and $(u_1^{(n+1,L-1)})_n$ have the same limit for $n \to \infty$, see Proposition 1, we subtract a suitable subsequence $(n_k)_k$ with limits $\eta_1^{(\infty)}, u_1^{(\infty)}$, and $\tilde{u}_2^{(\infty)}$ such that (6)-(8) respectively are still valid, cf. [9, Theorem 24.4, p 233], i.e., $0 \in \partial_H J^s(\cdot + \tilde{u}_2^{(\infty)}, u_1^{(\infty)} + \tilde{u}_2^{(\infty)})(u_1^{(\infty)}) + \eta_1^{(\infty)}$. By the definition of the subdifferential and Proposition 3 we obtain $J(u^{(\infty)}) = J^s(u^{(\infty)}, u^{(\infty)}) \leq J^s(v, u^{(\infty)}) + \langle \eta_1^{(\infty)}, u^{(\infty)} - v\rangle_H \leq J^s(v,u^{(\infty)}) + \|\eta_1^{(\infty)}\|_{\ell^2(\Omega)}\|u^{(\infty)} - v\|_{\ell^2(\Omega)}$ for all $v \in H$. Similarly one can show that $J(u^{(\infty)}) \leq J^s(v,u^{(\infty)}) + \|\eta_2^{(\infty)}\|_{\ell^2(\Omega)}\|u^{(\infty)} - v\|_{\ell^2(\Omega)}$ for all $v \in H$, and hence we have

$$J(u^{(\infty)}) \leq J^s(v,u^{(\infty)}) + \|\hat{\eta}\|_{\ell^2(\Omega)}\|u^{(\infty)} - v\|_{\ell^2(\Omega)} \tag{9}$$

for all $v \in H$, where $\|\hat{\eta}\|_{\ell^2(\Omega)} = \min\{\|\eta_1^{(\infty)}\|_{\ell^2(\Omega)}, \|\eta_2^{(\infty)}\|_{\ell^2(\Omega)}\}$.

Let $u^* \in \arg\min_{u\in H} J(u)$. Then there exists a $\rho \geq 0$ such that $J(u^{(\infty)}) = J(u^*) + \rho$.

1. If $\rho = 0$, then it immediately follows that $u^{(\infty)}$ is a minimizer of J.
2. If $\rho > 0$, then from the coercivity condition we obtain that there exists a constant $\beta > 0$, independent of α_T, such that $\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)} \le \beta < +\infty$.
3. If $\alpha_T < \frac{\gamma}{\beta^2\delta}$ for $0 < \gamma \le J(u^{(\infty)}) - J(u^*)$, then $J(u^{(\infty)}) \ge J(u^*) + \frac{\gamma}{\beta^2}\|u^{(\infty)} - u^*\|^2_{\ell^2(\Omega)}$. Setting $v = u^*$ in (9) and using the last inequality we obtain

$$\alpha_T\left(\delta\|u^* - u^{(\infty)}\|^2_{\ell^2(\Omega)} - \|T(u^* - u^{(\infty)})\|^2_{\ell^2(\Omega)}\right) + \|\hat{\eta}\|_{\ell^2(\Omega)}\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)} \ge \frac{\gamma}{\beta^2}\|u^{(\infty)} - u^*\|^2_{\ell^2(\Omega)}. \quad (10)$$

   From the latter inequality we get $\|\hat{\eta}\|_2 \ge (\frac{\gamma}{\beta^2} - \alpha_T\delta)\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)}$ and since $\alpha_T\delta < \frac{\gamma}{\beta^2}$ we obtain $\frac{\beta^2\|\hat{\eta}\|_{\ell^2(\Omega)}}{\gamma - \alpha_T\delta\beta^2} \ge \|u^{(\infty)} - u^*\|_{\ell^2(\Omega)}$.
4. If $\alpha_T > 0$ and T^*T is symmetric positive definite with smallest Eigenvalue $\sigma > 0$, then the factor $\frac{\gamma}{\beta^2}$ on the right hand side of the inequality in (10) is replaced by $\alpha_T\sigma$, cf. [6], and (10) reads as follows

$$\alpha_T(\sigma - \delta)\|u^* - u^{(\infty)}\|^2_{\ell^2(\Omega)} + \alpha_T\|T(u^* - u^{(\infty)})\|^2_{\ell^2(\Omega)} \le \|\hat{\eta}\|_{\ell^2(\Omega)}\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)}.$$

   By using once more the symmetric positive definiteness assumption on T^*T we obtain from the latter inequality that $\alpha_T(2\sigma - \delta)\|u^* - u^{(\infty)}\|^2_{\ell^2(\Omega)} \le \|\hat{\eta}\|_{\ell^2(\Omega)}\|u^{(\infty)} - u^*\|_{\ell^2(\Omega)}$. If $2\sigma > \delta$ then we get $\|u^* - u^{(\infty)}\|_{\ell^2(\Omega)} \le \frac{\|\hat{\eta}\|_{\ell^2(\Omega)}}{\alpha_T(2\sigma - \delta)}$.

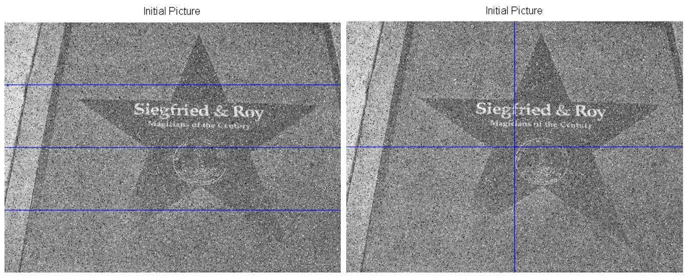
   □

4 Numerical Experiments

We present numerical experiments obtained by the parallel algorithm in (5) for the application of image deblurring, i.e., $S = T$ are blurring operators and $\varphi(|\nabla u|)(\Omega) = |\nabla u|(\Omega)$ (the total variation of u in Ω). The minimization problems of the subdomains are implemented in the same way as described in [6] by noting that the functional to be considered in each subdomain is now the strictly convex functional in (3).

We consider an image of size 1920×2576 pixels which is corrupted by Gaussian blur with kernel size 15×15 pixels and standard deviation 2. Additionally 4% salt-and-pepper noise (i.e., 4% of the pixels are either flipped to black or white) and Gaussian white noise with zero mean and variance 0.01 is added.

In order to show the efficiency of the parallel algorithm in (5) for decomposing the spatial domain into subomains, we compare its performance with the L^1-L^2-TV algorithm presented in [6], which solves the problem on all of Ω without any splitting. We consider splittings of the domain in stripes, cf. Figure 1(a), and in windows as depicted in Figure 1(b) for different numbers of subdomains ($D = 4, 16, 64$).



(a) (b)

Fig. 1 Image of size 1920×2576 pixels which is corrupted by Gaussian blur with kernel size 15×15 pixels and standard deviation 2, 4% salt-and-pepper noise, and Gaussian white noise with zero mean and variance 0.01. In (a) decomposition of the spatial domain into stripes and in (b) into windows.

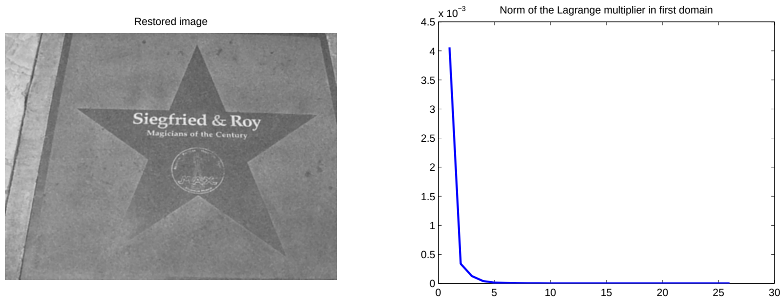
The algorithms are stopped as soon as the energy J reaches a significance level J^*, i.e., when $J(u^{(n)}) \leq J^*$ for the first time. For reason of comparison we experimentally choose $J^* = 0.059054$, i.e., we once restored the image of interest until we observed a visually satisfying restoration and the associated energy-value as J^*. In the subspace correction algorithm as well as in the L^1-L^2-TV algorithm we restore the image by setting $\alpha_S = 0.5$, $\alpha_T = 0.4$, and $\delta = 1.1$. The computations are done in Matlab on a computer with 256 cores and the multithreading-option is activated.

Table 1 presents the computational time and number of iterations the algorithms need to fulfill the stopping criterion for different number of subdomains. We clearly see that the domain decomposition algorithm for $D = 4, 16, 64$ subdomains is much faster than the L^1-L^2-TV algorithm ($D = 1$). Since a blurring operator is in general non-local, in each iteration $u^{(n)}$ has been communicated to each subdomain. Therefore the communication time becomes substantial for splittings into 16 or more domains such that the algorithm needs more time to reach the stopping criterion.

Table 1 Restoration of the image in Figure 1: Computational performance (CPU time in seconds and the number of iterations) for the global L^1-L^2-TV algorithm and for the parallel domain decomposition algorithms with $\alpha_1 = 0.5$, $\alpha_2 = 0.4$ for different numbers of subdomains ($D = 4, 16, 64$).

# Domains	window-splitting	stripe-splitting
$D = 1$ (L^1-L^2-TV alg.): 11944 s / 131 it		
$D = 4$:	2374 s / 27 it	2340 s / 27 it
$D = 16$:	2914 s / 27 it	2982 s / 27 it
$D = 64$:	7833 s / 27 it	8797 s / 28 it

In Figure 2 we depict the progress of the minimal Lagrange multiplier $\eta^{(n)} := \min_i\{\|\eta_i^{(n)}\|_{\ell^2(\Omega)}\}$, which indicates that the parallel algorithm indeed converges to a minimizer of the functional J.



(a) (b)

Fig. 2 (a) Restoration of the image in Figure 2 by the parallel subspace correction algorithm in (5). (b) The progress of the minimal Lagrange multiplier $\eta^{(n)}$.

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