A parallel preconditioner for a FETI-DP method for the Crouzeix-Raviart finite element

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1 Introduction

In this paper, we present a Neumann-Dirichlet type parallel preconditioner for a FETI-DP method for the nonconforming Crouzeix-Raviart (CR) finite element discretization of a model second order elliptic problem. The proposed method is almost optimal, in fact, the condition number of the preconditioned problem grows polylogarithmically with respect to the mesh parameters of the local triangulations.

In many scientific applications, where partial differential equations are used to model, the Crouzeix-Raviart (CR) finite element has been one of the most commonly used nonconforming finite element for the numerical solution. This includes applications like the Poisson equation (cf. [11, 23]), the Darcy-Stokes problem (cf. [8]), the elasticity problem (cf. [3]). We also would like to add that there is a close relationship between mixed finite elements and the nonconforming finite element for the second order elliptic problem; cf. [1, 2]. The CR element has also been used in the framework of finite volume element method; cf. [9].

There exists quite a number of works focusing on iterative methods for the CR finite element for second order problems; cf. [4, 5, 10, 13, 16, 18, 19, 20, 21, 22] and references therein. The purpose of this paper is to propose a parallel algorithm based on a Neumann-Dirichlet preconditioner for a FETI-DP formulation of the CR finite element method for the second order elliptic problem. To our knowledge, this is apparently the first work on such preconditioner for the FETI-DP method for the Crouzeix-Raviart (CR) finite element.

The FETI-DP method, which was first introduced in [12], describes a class of fast and efficient domain decomposition solvers for systems of algebraic equations arising from the finite element discretization of elliptic partial differential equations, cf. [17, 14, 15, 24] and references therein.

In a FETI-DP method one has to solve a linear system for a set of dual variables, formulated after eliminating the primal variables. The FETI-DP system contains in itself a coarse problem which is associated with the primal variables, while its preconditioner is based on solving only local problems which is fully parallel.

In this paper, we first present the Crouzeix Raviart discretization of the differential problem, a FETI-DP formulation of the problem is then introduced, and finally a Neumann-Dirichlet preconditioner for the FETI-DP problem is proposed.

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We present an almost optimal bound for the condition number, showing that the condition number of the preconditioned system grows like $C(1+\log(H/h))^2$, where H is the maximal diameter of the subdomains and h is the fine mesh size parameter.

2 Discrete problem

In this section we present the Crouzeix-Raviart finite element discretization of a model second order elliptic problem with discontinuous coefficients.

Let Ω be a polygonal domain in the plane. We assume that there exists a partition of Ω into disjoint polygonal subdomains Ω_k such that $\overline{\Omega} = \bigcup_{k=1}^N \overline{\Omega}_k$ with $\overline{\Omega}_k \cap \overline{\Omega}_l$ being an empty set, an edge or a vertex (crosspoint). We also assume that these subdomains form a coarse triangulation of the domain which is shape regular in the sense of [7]. We introduce a global interface $\Gamma = \bigcup_i \overline{\partial\Omega_i \setminus \partial\Omega}$ which plays an important role in our study.

Our model differential problem is to find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = \int_\Omega f v\, dx \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where $f \in L^2(\Omega)$, and $a(u,v) = \sum_{k=1}^N \int_{\Omega_k} \rho_k \nabla u \nabla v\, dx$. The coefficients ρ_k are positive and constant.

We assume that there exists a quasiuniform triangulation, $T_h = T_h(\Omega) = \{\tau\}$, of Ω such that any element τ of T_h is contained in only one subdomain, as a consequence any subdomain Ω_k inherits a local triangulation $T_h(\Omega_k) = \{\tau\}_{\tau\subset\Omega_k, \tau\in T_h}$.

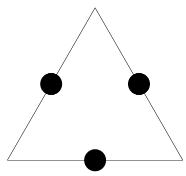

Fig. 1 Illustrating the CR finite element in 2D with black dots as the CR nodal points or CR nodes.

Let $h = \max_{\tau\in T_h(\Omega)} \text{diam}(\tau)$ be the mesh size parameter of the triangulation, cf. [6]. We introduce the following sets of Crouzeix-Raviart (CR) nodal points or - nodes: $\Omega_h^{CR}, \partial\Omega_h^{CR}, \Omega_{k,h}^{CR}, \partial\Omega_{k,h}^{CR}$, and $\Gamma_{kl,h}^{CR}$ correspond to $\Omega, \partial\Omega, \Omega_k, \partial\Omega_k$, and Γ_{kl}, respectively. Here Γ_{kl} is an interface, an open edge, which is shared by the two subdomains, Ω_k and Ω_l.

We now introduce the local finite element spaces. Let $\widehat{W}^h(\Omega)$ be the Crouzeix-Raviart finite element space defined as follows,

$$\widehat{W}^h(\Omega) = \{u \in L^2(\Omega) : u|_\tau \in P_1(\tau) \text{ for each triangle } \tau \in T_h(\Omega), \\ u \text{ is continuous at every midpoint } m \in \Omega_h^{CR} \tag{2}$$

and $u(m) = 0$ for every $m \in \partial\Omega_h^{CR}\}$.

Here $P_1(\tau)$ is the function space of linear polynomials defined over τ. The degrees of freedom of a function $u \in \widehat{W}^h(\Omega)$ over $\tau \in T_h(\Omega)$ are: $\{u(m_j)\}_{j=1,2,3}$, where m_j is a midpoint of an edge of τ, cf. Fig. 1.

We define the local CR space $W^h(\Omega_k)$ as the space of functions which are restrictions to Ω_k of the functions of $\widehat{W}^h(\Omega)$, i.e. $W^h(\Omega_k) = \{u_{|\Omega_k} : u \in \widehat{W}^h(\Omega)\}$. The standard nodal basis function, ϕ_x^{CR}, of $W^h(\Omega_k)$, associated with the CR nodal point $x \in \overline{\Omega}_k^{CR}$, is a function which is equal to one at x and zero at the remaining CR nodal points of $\overline{\Omega}_k^{CR} \setminus \partial\Omega^{CR}$. $\{\phi_x^{CR}\}_{x\in\overline{\Omega}_k^{CR}\setminus\partial\Omega^{CR}}$ is the standard nodal basis of $W^h(\Omega_k)$.

The discrete problem is then defined as follows: Find $u_h^* \in \widehat{W}^h(\Omega)$ such that

$$a_h(u_h^*, v) = f(v) \quad \forall v \in \widehat{W}^h(\Omega), \tag{3}$$

where $a_h(u,v) := \sum_{k=1}^N a_{k,h}(u,v)$ with the local broken bilinear form:

$$a_{k,h}(u,v) := \sum_{\tau\in T_h(\Omega_k)} \int_\tau \rho_k \nabla u \nabla v \, dx.$$

This problem has a unique solution, and an optimal error bound is known; cf. [6].

We shall now reformulate (3) as a saddle point problem. We start by introducing the following global space defined over Ω as follows,

$$W^h(\Omega) := \Pi_{k=1}^N W^h(\Omega_k).$$

Note that each interface Γ_{kl} inherits a 1D triangulation $T_h(\Gamma_{kl})$ from T_h. We define $V^h(\Gamma_{kl})$ as the space of piecewise constant functions over $T_h(\Gamma_{kl})$. In FETI-DP, an important role is played by the global interface which is defined as $\overline{\Gamma} := \bigcup_{k=1}^N \partial\Omega_k \setminus \partial\Omega$. Then, let

$$V^h(\Gamma) := \prod_{\Gamma_{kl}\subset\Gamma} V^h(\Gamma_{kl})$$

be the auxiliary interface space which will be later used as the space of Lagrange multipliers. We introduce the bilinear form $b(u,\psi) : W^h(\Omega) \times V^h(\Gamma) \to \mathbb{R}$ as follows: let $u = (u_k)_{k=1}^N \in W^h(\Omega)$ and $\psi = (\psi_{lk})_{\Gamma_{kl}} \in V^h(\Gamma)$, then $b(u,\psi) = \sum_{\Gamma_{kl}\subset\Gamma} b_{lk}(u,\psi_{lk})$ with

$$b_{lk}(u,\psi_{lk}) = \int_{\Gamma_{kl}} (u_k - u_l)\psi_{lk} \, ds \qquad k > l.$$

Throughout the rest of this paper, we will use the same notation to denote a function and its vector representation with values of the degrees of freedom (dofs) of this function as entries in the representation.

Let c_r be a crosspoint, which is a subdomain vertex, not lying on $\partial\Omega$, and let $\mathscr{V}^{CR}(c_r)$ be the set of CR nodal points of those triangle edges that lie on subdomain boundaries and are incident to c_R, e.g. the black dots in Figure 2. Let $\mathscr{V}^{CR} = \bigcup_{c_r \in \Gamma} \mathscr{V}^{CR}(c_r)$.

We then introduce $\widetilde{W}^h(\Omega)$ as the subspace of $W^h(\Omega)$ of functions which are continuous at the CR nodes of $\mathscr{V}^{CR}$. We also introduce a reduced Lagrange multiplier space as follows,

$$\widetilde{V}^h(\Gamma) := \{\lambda \in V^h(\Gamma) : \lambda(m) = 0 \quad \forall m \in \Gamma_h^{CR} \cap \mathscr{V}^{CR}\} \subset V^h(\Gamma).$$

The discrete problem can now be reformulated as the following saddle point prob-

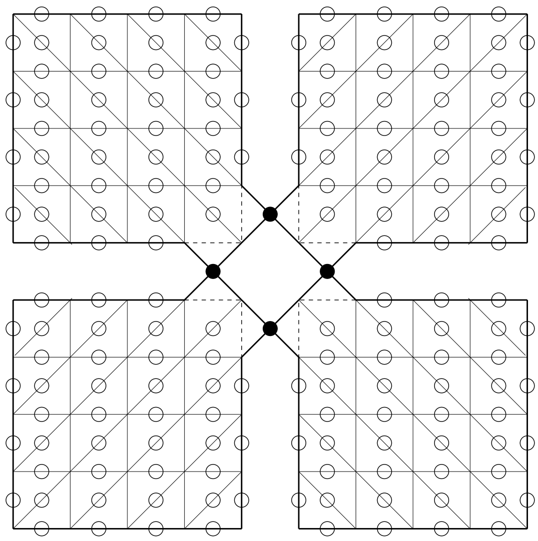

Fig. 2 Illustrating a four subdomain case with one crosspoint. Black dots in the figure represent the CR nodes of $\mathscr{V}^{CR}$ corresponding to the cross point. CR nodes (both circles and black dots) which the degrees of freedom (dofs) of $\widetilde{W}^h(\Omega)$ are associated with, are also shown.

lem: find the pair $(u_h^*, \lambda^*) \in \widetilde{W}^h(\Omega) \times \widetilde{V}^h(\Gamma)$ such that

$$\begin{aligned} a(u_h^*, v) + b(v, \lambda^*) &= f(v) \quad \forall v \in \widetilde{W}^h(\Omega), \\ b(u_h^*, \phi) &= 0 \qquad \forall \phi \in \widetilde{V}^h(\Gamma). \end{aligned} \tag{4}$$

Any vector w corresponding to the function $w \in \widetilde{W}^h(\Omega)$ (note that we are using the same symbol for the function and its vector representation) can be decomposed as follows,

$$w = (w^{(i)}, w^{(c)}, w^{(r)}),$$

where $w^{(i)}$ is the vector with dofs associated with the CR nodes of the subdomain interior, $w^{(c)}$ is the vector with dofs associated with the CR nodes of $\mathscr{V}^{CR}$, and $w^{(r)}$ is the vector with dofs associated with the remaining dofs.

Analogously, let $W \subset \widetilde{W}^h(\Omega)$ be the space corresponding to the vectors with the dofs associated with Γ, then we can decompose any vector w of $w \in W$ as $w = (w^{(c)}, w^{(r)})$.

Now let $W_r = \{w^{(r)} : w \in \widetilde{W}^h(\Omega)\}$, in other words, W_r is the space of functions representing the dofs associated with the CR nodes on Γ, not belonging to the set $\mathscr{V}^{CR}$.

Note that $w^{(r)} \in W_r$ has two degrees of freedom associated with each midpoint on $\Gamma \setminus \mathscr{V}^{CR}$, for instance, if $m \in \Gamma^{CR}_{kl,h}$ then its associated two degrees of freedom are $w_k(m)$ and $w_l(m)$.

We introduce A as a block diagonal matrix with local stiffness matrices as the blocks, i.e., $A := \mathrm{diag}(A_k)_{k=1}^N$ with A_k being the stiffness matrix generated by $a_{k,h}(\cdot,\cdot)$ in the standard nodal basis of $W^h(\Omega_k)$.

Let $B = \mathrm{diag}(B^{(kl)})_{\Gamma_{kl}}$ be a block diagonal matrix with $B^{(kl)}$ related to the edge $\Gamma_{kl} \subset \Gamma$ (for $k > l$) containing only zeros, ones and minus ones as matrix entries, and w_h^* is the vector representation of the function $w_h^* \in W$ (denoted by the same symbol).

We note that each block A_j associated with an inner subdomain Ω_j (subdomain not having an edge on $\partial\Omega$), is singular and therefore cannot be inverted. As part of our FETI-DP algorithm, we enforce continuity at the CR nodes close to the crosspoints, i.e., at the CR nodes of $\mathscr{V}^{CR}$, thereby remove the problem of noninvertibility.

We introduce the Schur complement matrix, S, of A, with respect to the unknowns associated with Γ, which is obtained after eliminating the unknowns associated with the subdomain interior. We note that S is a block diagonal matrix.

3 FETI-DP problem

Let $\tilde{A}$ be the matrix obtained from block diagonal matrix A by taking into account the continuity of the degrees of freedom at $\mathscr{V}^{CR}$. Let $\tilde{A}$ be partitioned into

$$\tilde{A} = \begin{pmatrix} A_{ii} & A_{ic} & A_{ir} \\ A_{ci} & A_{cc} & A_{cr} \\ A_{ri} & A_{rc} & A_{rr} \end{pmatrix},$$

where the subscript i and superscript (i) refer to the dofs associated with CR nodes in the subdomain interior, the subscript c and superscript (c) to the dofs associated with the crosspoints, and the subscript r and superscript (r) to the dofs associated with the remaining CR nodes.

The matrix formulation of (4) takes the following form,

$$\begin{pmatrix} A_{ii} & A_{ic} & A_{ir} & 0 \\ A_{ci} & A_{cc} & A_{cr} & 0 \\ A_{ri} & A_{rc} & A_{rr} & (B^{(r)})^T \\ 0 & 0 & B^{(r)} & 0 \end{pmatrix} \begin{pmatrix} u^{(i)} \\ u^{(c)} \\ u^{(r)} \\ \lambda^* \end{pmatrix} = \begin{pmatrix} f_i \\ f_c \\ f_r \\ 0 \end{pmatrix}, \tag{5}$$

where $B^{(r)}$ is the submatrix of B, associated with the CR nodes that are on Γ but not in $\mathscr{V}^{CR}$.

Eliminating the unknowns corresponding to the subdomain interior CR nodes and the crosspoints, i.e., $u^{(i)}$ and $u^{(c)}$, in (5) we arrive at

$$\begin{array}{ll} \tilde{S}u^{(r)} + (B^{(r)})^T\lambda^* &= \tilde{f}_r, \\ B^{(r)}u^{(r)} &= 0, \end{array} \tag{6}$$

where $\tilde{S} = A_{rr} - \begin{pmatrix} A_{ri} & A_{rc} \end{pmatrix} \begin{pmatrix} A_{ii} & A_{ic} \\ A_{ci} & A_{cc} \end{pmatrix}^{-1} \begin{pmatrix} A_{ir} \\ A_{cr} \end{pmatrix}$.

Further eliminating $u^{(r)}$, we obtain the following FETI-DP problem: find $\lambda^* \in M$ such that

$$F(\lambda^*) = d, \tag{7}$$

where $d := -B^{(r)}\tilde{S}^{-1}\tilde{f}_r$ and $F := B^{(r)}\tilde{S}^{-1}(B^{(r)})^T$.

4 Parallel preconditioner

The general idea of our Neumann-Dirichlet preconditioner for the FETI-DP system comes from [14], where the case of nonmatching grids and standard continuous P_1 finite element were considered.

We start by further decomposing the vector $w^{(r)}$ into its two component vectors, i.e.,

$$w^{(r)} = \left(w_\Gamma^{(r)}, w_\Delta^{(r)}\right)^T,$$

where $w_\Gamma^{(r)} = (w_{kl,\Gamma}^{(r)})_{\Gamma_{kl}}$ with

$$w_{kl,\Gamma}^{(r)}(m) = \begin{cases} w_k^{(r)}(m) & if\ \rho_k > \rho_l \\ w_k^{(r)}(m) & if\ \rho_k = \rho_l, \quad k > l, \\ w_l^{(r)}(m) & otherwise \end{cases} \qquad m \in \Gamma_{kl,h}^{CR}$$

i.e., $w_{kl,\Gamma}^{(r)}$ is the vector with those entries of $w^{(r)}$ which are related to Γ_{kl} and to the subdomain Ω_s with the larger coefficient ρ_s, $s = k,l$. In case of equality we pick the ones related to Ω_k with $k > l$. The vector $w_\Delta^{(r)}$ corresponds to the remaining dofs of $w^{(r)}$. Correspondingly, we introduce $W_\Delta = \{w_\Delta^{(r)} : w^{(r)} \in W_r\}$, which is a subspace of W_r, consisting of functions which are defined by the values at the CR nodes on the interface Γ_{kl} belonging to the subdomain Ω_s, $s = k,l$, with the smaller coefficient. We note that dim $\widetilde{V}^h(\Gamma) =$ dim W_Δ, which equals the number of CR nodes on $\Gamma \setminus \mathscr{V}^{CR}$.

Let S_Δ be the matrix obtained by restricting the block diagonal Schur complement matrix $S : W \to W$ to W_Δ. Note that this matrix can be represented as a block diagonal matrix with nonsingular diagonal blocks $S_{k,\Delta}$, i.e.

$$S_\Delta := \mathrm{diag}(S_{k,\Delta})_k,$$

where the subscript k runs over the subdomains Ω_k such that $S_{k,\Delta}$ correspond to the CR nodes of $\partial\Omega_k^{CR}$ and these CR nodes which are dofs of $w \in W_\Delta$.

We define the nonsingular block diagonal matrix $B_\Delta : W_\Delta \to W_\Delta$, as

$$B_\Delta := \mathrm{diag}(B^{(r)}_{\Delta,\Gamma_{kl}})_{\Gamma_{kl}\subset\Gamma},$$

where $B^{(r)}_{\Delta,\Gamma_{kl}}$ is a diagonal block of the matrix $B^{(r)}$, corresponding to Γ_{kl} and these CR nodes which are dofs of $w \in W_\Delta$. Note that these blocks are nonsingular.

The parallel preconditioner is then as follows,

$$\mathscr{M}_{DN}^{-1} := B_\Delta^{-T} S_\Delta B_\Delta^{-1},$$

which is nonsingular, and its inverse is $\mathscr{M}_{DN} := B_\Delta S_\Delta^{-1} B_\Delta^T$.

5 Condition number bounds

The main result of this paper is the following theorem which yields a bound for the condition number of the preconditioned system.

Theorem 1 (Condition number estimate). *It holds that*

$$\langle \mathscr{M}_{DN}\lambda,\lambda\rangle \le \langle F\lambda,\lambda\rangle \le C\left(1+\log\left(\frac{H}{h}\right)\right)^2 \langle \mathscr{M}_{DN}\lambda,\lambda\rangle \qquad \forall\lambda \in M,$$

where $H = \max_k \mathrm{diam}(\Omega_k)$ and C is a positive constant independent of the coefficients, and the mesh size parameters H and h. Here $\langle\cdot,\cdot\rangle$ is the standard l_2 inner product.

As a direct consequence of this theorem, we see that the condition number of the preconditioned matrix $\mathscr{M}_{DN}^{-1}F$ is bounded by $C\left(1+\log\left(\frac{H}{h}\right)\right)^2$.

The lower bound in the theorem is obtained by a purely algebraic argument, while we get the upper bound by using several technical results of which the most important one is the estimate of special trace norms of jumps of tangential and normal traces over the interface $\Gamma_{kl} \subset \Gamma$.

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