Domain decomposition methods for problems of unilateral contact between elastic bodies with nonlinear Winkler covers*

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1 Introduction

Thin covers from different materials are often applied in engineering to improve the functional properties of the surfaces of machines and structures components. On the other hand, thin covers with certain mechanical properties are used to model the real microstructure of surfaces, adhesion and glue bondings [6, 14, 15].

The classical methods for solution of contact problems for bodies with thin covers are grounded on integral equations and are reviewed in work [15]. Nowadays, one of the most effective numerical methods for such contact problems are methods, based on variational formulations and finite element approximations.

Efficient approach for solution of multibody contact problems is the use of domain decomposition methods (DDMs). Many DDMs for contact problems without covers are obtained on discrete level [3, 16]. Among DDMs, proposed on continuous level for contact problems without covers are methods presented in [1, 9, 12]. Domain decomposition methods for solution of problem of ideal contact between two bodies, connected through nonlinear Winkler layer are proposed in [2, 8]. These methods are based on saddle-point formulation and conjugate gradient methods.

In current contribution we consider a problem of unilateral contact between elastic bodies with nonlinear Winkler covers. We give variational formulations of this problem in the form of nonquadratic variational inequality on convex set and nonlinear variational equation in the whole space, and present theorems about existence and uniqueness of their solution. Furthermore, we propose on continuous level a class of parallel domain decomposition methods for solving the nonlinear variational equation, which corresponds to original contact problem. In each iteration of these methods we have to solve in a parallel way linear variational equations in separate bodies, which are equivalent in a weak sense to linear elasticity problems with Robin boundary conditions on possible contact areas. These DDMs are based on abstract nonstationary iterative methods for variational equations in Banach spaces. They are the generalization of domain decomposition methods, proposed by us earlier in [4, 5, 10] for unilateral contact problems without covers. Some particular cases of proposed DDMs can be viewed as a modification of semismooth Newton method [7]. The numerical analysis of obtained DDMs is made for plane contact problems using finite element approximations.

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2 Statement of the problem

Consider a unilateral contact of N elastic bodies $\Omega_\alpha \subset \mathbb{R}^3$ with sufficiently smooth boundaries Γ_α, $\alpha = 1,2,...,N$ (Fig.1a). Suppose that across each contact surface there is a nonlinear Winkler layer. Denote $\Omega = \bigcup_{\alpha=1}^{N} \Omega_\alpha$.

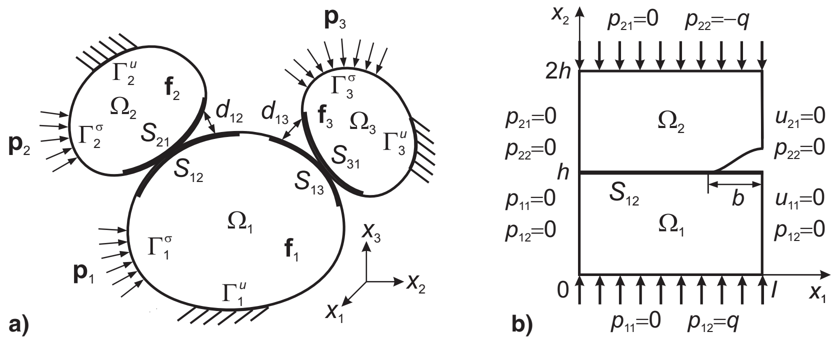

Fig. 1 Unilateral contact between several elastic bodies through nonlinear Winkler layers

A stress-strain state in point $\mathbf{x} = (x_1, x_2, x_3)^\top$ of each solid Ω_α is described by the displacement vector $\mathbf{u}_\alpha = u_{\alpha i}\,\mathbf{e}_i$, the tensor of strains $\hat{\boldsymbol{\varepsilon}}_\alpha = \varepsilon_{\alpha ij}\,\mathbf{e}_i\,\mathbf{e}_j$ and the tensor of stresses $\hat{\boldsymbol{\sigma}}_\alpha = \sigma_{\alpha ij}\,\mathbf{e}_i\,\mathbf{e}_j$. These quantities satisfy the following relations:

$$\sum_{j=1}^{3} \frac{\partial \sigma_{\alpha ij}(\mathbf{x})}{\partial x_j} + f_{\alpha i}(\mathbf{x}) = 0\,, \;\; \mathbf{x} \in \Omega_\alpha\,, \;\; i = 1,2,3\,, \tag{1}$$

$$\sigma_{\alpha ij}(\mathbf{x}) = \sum_{k,l=1}^{3} C_{\alpha ijkl}(\mathbf{x})\,\varepsilon_{\alpha kl}(\mathbf{x})\,, \;\; \varepsilon_{\alpha ij} = \frac{1}{2}\left(\frac{\partial u_{\alpha i}}{\partial x_j} + \frac{\partial u_{\alpha j}}{\partial x_i}\right), \;\; i,j = 1,2,3\,, \tag{2}$$

where $f_{\alpha i}$ are the components of volume forces vector $\mathbf{f}_\alpha = f_{\alpha i}\,\mathbf{e}_i$, and $C_{\alpha ijkl}$ are symmetric elasticity constants, which are bounded in the following sense:

$$(\exists b_\alpha, c_\alpha > 0)\;(\forall \mathbf{x})\;\left\{ b_\alpha \sum_{i,j=1}^{3} \varepsilon_{\alpha ij}^2 \le \sum_{i,j,k,l=1}^{3} C_{\alpha ijkl}\,\varepsilon_{\alpha ij}\,\varepsilon_{\alpha kl} \le c_\alpha \sum_{k,l=1}^{3} \varepsilon_{\alpha kl}^2 \right\}. \tag{3}$$

On the boundary Γ_α introduce a local orthonormal coordinate system $\boldsymbol{\xi}_\alpha$, $\boldsymbol{\eta}_\alpha$, $\mathbf{n}_\alpha$, where $\mathbf{n}_\alpha$ is an outer unit normal, and $\boldsymbol{\xi}_\alpha$, $\boldsymbol{\eta}_\alpha$ are unit tangents. Then the vectors of displacements and stresses on Γ_α can be written in the following way: $\mathbf{u}_\alpha = u_{\alpha\xi}\,\boldsymbol{\xi}_\alpha + u_{\alpha\eta}\,\boldsymbol{\eta}_\alpha + u_{\alpha n}\,\mathbf{n}_\alpha$, $\boldsymbol{\sigma}_\alpha = \hat{\boldsymbol{\sigma}}_\alpha \cdot \mathbf{n}_\alpha = \sigma_{\alpha\xi}\,\boldsymbol{\xi}_\alpha + \sigma_{\alpha\eta}\,\boldsymbol{\eta}_\alpha + \sigma_{\alpha n}\,\mathbf{n}_\alpha$.

Suppose, that the boundary Γ_α consists of three disjoint parts: $\Gamma_\alpha = \Gamma_\alpha^u \bigcup \Gamma_\alpha^\sigma \bigcup S_\alpha$, $\Gamma_\alpha^u = \overline{\Gamma_\alpha^u}$, $\Gamma_\alpha^u \neq \emptyset$, $S_\alpha \neq \emptyset$. On the part Γ_α^u homogenous Dirichlet boundary conditions are prescribed, and on the part Γ_α^σ we consider Neumann boundary conditions:

$$\mathbf{u}_\alpha(\mathbf{x}) = 0, \;\; \mathbf{x} \in \Gamma_\alpha^u\,; \;\; \boldsymbol{\sigma}_\alpha(\mathbf{x}) = \mathbf{p}_\alpha(\mathbf{x}), \;\; \mathbf{x} \in \Gamma_\alpha^\sigma\,. \tag{4}$$

The part $S_\alpha = \bigcup_{\beta\in B_\alpha} S_{\alpha\beta}$, $\bigcap_{\beta\in B_\alpha} S_{\alpha\beta} = \emptyset$ is the possible contact area of body Ω_α with the other bodies. Here $S_{\alpha\beta}$ is the possible unilateral contact area of body Ω_α with body Ω_β, and $B_\alpha \subset \{1,2,...,N\}$ is the set of the indices of all bodies in contact with body Ω_α. We assume that the surfaces $S_{\alpha\beta} \subset \Gamma_\alpha$ and $S_{\beta\alpha} \subset \Gamma_\beta$ are sufficiently close ($S_{\alpha\beta} \approx S_{\beta\alpha}$), and $\mathbf{n}_\alpha(\mathbf{x}) \approx -\mathbf{n}_\beta(\mathbf{x}')$, $\mathbf{x} \in S_{\alpha\beta}$, $\mathbf{x}' = P(\mathbf{x}) \in S_{\beta\alpha}$, where $P(\mathbf{x})$ is the projection of point $\mathbf{x}$ on $S_{\alpha\beta}$. Let $d_{\alpha\beta}(\mathbf{x}) = \pm\|\mathbf{x}-\mathbf{x}'\|$ be a distance between bodies Ω_α and Ω_β before the deformation. We suppose that possible contact areas $S_{\alpha\beta}$ and $S_{\beta\alpha}$, $\beta \in B_\alpha$, $\alpha = 1,...,N$ have nonlinear Winkler covers. Total compression $w_{\alpha\beta}$ of these covers is related with normal contact stress as follows: $\sigma_{\alpha n}(\mathbf{x}) = \sigma_{\beta n}(\mathbf{x}') = g_{\alpha\beta}\left(w_{\alpha\beta}(\mathbf{x})\right)$, $\mathbf{x} \in S_{\alpha\beta}$, $\mathbf{x}' \in S_{\beta\alpha}$, where $g_{\alpha\beta}$ is given nonlinear continuous function, which satisfies the following conditions:

$$g_{\alpha\beta}(0) = 0\,,\quad (\forall y,z)\left\{\, y<z \Rightarrow g_{\alpha\beta}(y) < g_{\alpha\beta}(z)\,\right\}, \tag{5}$$

$$\left(\exists M_{\alpha\beta} > 0\right)\,(\forall y,z)\,\left\{\,\left|g_{\alpha\beta}(y) - g_{\alpha\beta}(z)\right| \le M_{\alpha\beta}\,|y-z|\,\right\}. \tag{6}$$

On possible contact zones $S_{\alpha\beta}$, $\beta \in B_\alpha$, $\alpha = 1,2,...,N$ we consider the following unilateral contact conditions through nonlinear Winkler layers:

$$\sigma_{\alpha\xi}(\mathbf{x}) = \sigma_{\beta\xi}(\mathbf{x}') = 0\,,\;\; \sigma_{\alpha\eta}(\mathbf{x}) = \sigma_{\beta\eta}(\mathbf{x}') = 0\,, \tag{7}$$

$$\sigma_{\alpha n}(\mathbf{x}) = \sigma_{\beta n}(\mathbf{x}') = g_{\alpha\beta}\left(w_{\alpha\beta}(\mathbf{x})\right) \le 0,\;\; u_{\alpha n}(\mathbf{x}) + u_{\beta n}(\mathbf{x}') + w_{\alpha\beta}(\mathbf{x}) \le d_{\alpha\beta}(\mathbf{x}), \tag{8}$$

$$\left[u_{\alpha n}(\mathbf{x}) + u_{\beta n}(\mathbf{x}') + w_{\alpha\beta}(\mathbf{x}) - d_{\alpha\beta}(\mathbf{x})\right]\sigma_{\alpha n}(\mathbf{x}) = 0\,,\;\; \mathbf{x}' = P(\mathbf{x})\,,\;\; \mathbf{x} \in S_{\alpha\beta}\,. \tag{9}$$

3 Variational formulations

For each body Ω_α consider Sobolev space $V_\alpha = [H^1(\Omega_\alpha)]^3$ and the closed subspace $V_\alpha^0 = \{\mathbf{u}_\alpha \in V_\alpha :\; \mathbf{u}_\alpha = 0 \text{ on } \Gamma_\alpha^u\}$. All values of the elements from these spaces on the parts of boundary Γ_α should be understood as traces. The trace of element $\mathbf{u}_\alpha \in V_\alpha$ on the part Γ_α^u should belong to space $[H^{1/2}(\Gamma_\alpha^u)]^3$, and the trace of element from V_α^0 on the part $\Xi_\alpha = \mathrm{int}\,(\Gamma_\alpha \setminus \Gamma_\alpha^u)$ should belong to $[H_{00}^{1/2}(\Xi_\alpha)]^3$.

Define Hilbert space $V_0 = \prod_{\alpha=1}^N V_\alpha$ with scalar product $(\mathbf{u}\,,\mathbf{v})_{V_0} = \sum_{\alpha=1}^N (\mathbf{u}_\alpha, \mathbf{v}_\alpha)_{V_\alpha}$ and norm $\|\mathbf{u}\|_{V_0} = (\mathbf{u}\,,\mathbf{u})_{V_0}^{1/2}$, $\mathbf{u},\mathbf{v} \in V_0$. Moreover, introduce the following spaces $W = \{\mathbf{w} = (w_{\alpha\beta})^\top_{\{\alpha,\beta\}\in Q} :\; w_{\alpha\beta} \in H_{00}^{1/2}(\Xi_\alpha)\}$ and $U_0 = V_0 \times W = \{\mathbf{U} = (\mathbf{u},\mathbf{w})^\top :\; \mathbf{u} \in V_0, \mathbf{w} \in W\}$, where $Q = \{\{\alpha,\beta\} :\; \alpha \in \{1,2,...,N\}\,,\; \beta \in B_\alpha\}$.

In space U_0 consider the closed convex set of all displacements, which satisfy nonpenentration contact conditions: $K = \{\mathbf{U} \in U_0 :\; u_{\alpha n} + u_{\beta n} + w_{\alpha\beta} \le d_{\alpha\beta}$ on $S_{\alpha\beta}$, $\{\alpha,\beta\} \in Q\}$, where $u_{\alpha n} = \mathbf{n}_\alpha \cdot \mathbf{u}_\alpha \in H_{00}^{1/2}(\Xi_\alpha)$, $w_{\alpha\beta}, d_{\alpha\beta} \in H_{00}^{1/2}(\Xi_\alpha)$.

Let us introduce bilinear form $A(\mathbf{u},\mathbf{v}) = \sum_{\alpha=1}^N a_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha)$, $\mathbf{u},\mathbf{v} \in V_0$, $a_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha) = \int_{\Omega_\alpha} \hat{\boldsymbol{\sigma}}_\alpha(\mathbf{u}_\alpha) : \hat{\boldsymbol{\varepsilon}}_\alpha(\mathbf{v}_\alpha)\, d\Omega$, such that $A(\mathbf{u},\mathbf{u})$ represents the total elastic deformation energy of the bodies, linear form $L(\mathbf{u}) = \sum_{\alpha=1}^N l_\alpha(\mathbf{u}_\alpha)$, $l_\alpha(\mathbf{u}_\alpha) = \int_{\Omega_\alpha} \mathbf{f}_\alpha \cdot \mathbf{u}_\alpha\, d\Omega\, +$

$\int_{\Gamma_\alpha^\sigma} \mathbf{p}_\alpha \cdot \mathbf{u}_\alpha \, dS$, $\mathbf{f}_\alpha \in [L_2(\Omega_\alpha)]^3$, $\mathbf{p}_\alpha \in [H_{00}^{-1/2}(\Xi_\alpha)]^3$, which is equal to external forces work, and nonquadratic functional $H(\mathbf{w}) = \sum_{\{\alpha,\beta\}\in Q} \int_{S_{\alpha\beta}} \left[\int_0^{w_{\alpha\beta}} g_{\alpha\beta}(z)\, dz \right] dS$, $\mathbf{w} \in W$, which represents the total deformation energy of nonlinear Winkler layers.

We have shown, that bilinear form A is symmetric, continuous and coercive if condition (3) holds, and nonquadratic functional H is Gateaux differentiable: $H'(\mathbf{w},\mathbf{z}) = \sum_{\{\alpha,\beta\}\in Q} \int_{S_{\alpha\beta}} g_{\alpha\beta}(w_{\alpha\beta})\, z_{\alpha\beta}\, dS$, $\mathbf{w},\mathbf{z} \in W$.

Theorem 1. *Suppose that conditions (3), (5), (6) hold. Then problem (1), (2), (4), (7)–(9) has an alternative weak formulation as the following minimization problem:*

$$F(\mathbf{U}) = A(\mathbf{u},\mathbf{u})/2 - L(\mathbf{u}) + H(\mathbf{w}) \to \min_{\mathbf{U}\in K} . \tag{10}$$

Moreover, there exists a unique solution of problem (10), and this problem is equivalent to the following nonquadratic variational inequality on set K:

$$F'(\mathbf{U},\mathbf{V}-\mathbf{U}) = A(\mathbf{u},\mathbf{v}-\mathbf{u}) - L(\mathbf{v}-\mathbf{u}) + H'(\mathbf{w},\mathbf{z}-\mathbf{w}) \geq 0, \;\; \forall (\mathbf{v},\mathbf{z})^\top \in K . \tag{11}$$

Except this variational formulation, we also have proposed another weak formulation of original contact problem in the form of nonlinear variational equation.

Let us introduce the following nonquadratic functional in space V_0:

$$J(\mathbf{u}) = \sum_{\{\alpha,\beta\}\in Q} \int_{S_{\alpha\beta}} \left[\int_0^{d_{\alpha\beta} - u_{\alpha n} - u_{\beta n}} g_{\alpha\beta}^-(z)\, dz \right] dS, \;\; \mathbf{u} \in V_0 , \tag{12}$$

where $g_{\alpha\beta}^-(z) = \{ 0 ,\, z \geq 0 \} \vee \{ g_{\alpha\beta}(z) ,\, z < 0 \}$ is nonlinear function.

Functional $J(\mathbf{u})$ is nonnegative and Gateaux differentiable in V_0: $J'(\mathbf{u},\mathbf{v}) = -\sum_{\{\alpha,\beta\}\in Q} \int_{S_{\alpha\beta}} g_{\alpha\beta}^-(d_{\alpha\beta} - u_{\alpha n} - u_{\beta n})\, [v_{\alpha n} + v_{\beta n}]\, dS$. We have shown that if conditions (5) and (6) hold, then Gateaux differential $J'(\mathbf{u},\mathbf{v})$ satisfies the following properties: $(\forall \mathbf{u} \in V_0)\, (\exists \tilde{R} > 0)\, (\forall \mathbf{v} \in V_0) \{ |J'(\mathbf{u},\mathbf{v})| \leq \tilde{R} \|\mathbf{v}\|_{V_0} \}$, $(\exists \tilde{D} > 0)\, (\forall \mathbf{u},\mathbf{v},\mathbf{w} \in V_0) \{ |J'(\mathbf{u}+\mathbf{w},\mathbf{v}) - J'(\mathbf{u},\mathbf{v})| \leq \tilde{D} \|\mathbf{v}\|_{V_0} \|\mathbf{w}\|_{V_0} \}$, $(\forall \mathbf{u},\mathbf{v} \in V_0)$ $\{ J'(\mathbf{u}+\mathbf{v},\mathbf{v}) - J'(\mathbf{u},\mathbf{v}) \geq 0 \}$. These properties helped us to prove the next theorem.

Theorem 2. *Suppose that conditions (3), (5) and (6) hold. Then the contact problem (1), (2), (4), (7)–(9) is equivalent to problem (1), (2), (4), (7) with the following nonlinear boundary value conditions on the possible contact areas:*

$$\sigma_{\alpha n}(\mathbf{x}) = \sigma_{\beta n}(\mathbf{x}') = g_{\alpha\beta}^- \left(d_{\alpha\beta}(\mathbf{x}) - u_{\alpha n}(\mathbf{x}) - u_{\beta n}(\mathbf{x}') \right), \;\; \mathbf{x}' = P(\mathbf{x}), \;\; \mathbf{x} \in S_{\alpha\beta} , \tag{13}$$

and it is equivalent in weak sense to the next nonquadratic minimization problem:

$$F_1(\mathbf{u}) = A(\mathbf{u},\mathbf{u})/2 - L(\mathbf{u}) + J(\mathbf{u}) \to \min_{\mathbf{u}\in V_0} . \tag{14}$$

Moreover, problem (14) has a unique solution and is equivalent to the following nonlinear variational equation in space V_0:

$$F_1'(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) + J'(\mathbf{u},\mathbf{v}) - L(\mathbf{v}) = 0, \;\; \forall \mathbf{v} \in V_0 , \;\; \mathbf{u} \in V_0 . \tag{15}$$

4 Nonstationary iterative methods

In reflexive Banach space V consider an abstract nonlinear variational equation

$$\Phi(\mathbf{u},\mathbf{v}) = Y(\mathbf{v}), \ \ \forall \mathbf{v} \in V, \ \ \mathbf{u} \in V, \tag{16}$$

where $\Phi : V \times V \to \mathbb{R}$ is a functional, which is linear in $\mathbf{v}$, but nonlinear in $\mathbf{u}$, and $Y : V \to \mathbb{R}$ is linear continuous form. For numerical solution of (16) consider the following nonstationary iterative method [5, 11]:

$$G^k(\mathbf{u}^{k+1},\mathbf{v}) = G^k(\mathbf{u}^k,\mathbf{v}) - \gamma^k \left[\Phi(\mathbf{u}^k,\mathbf{v}) - Y(\mathbf{v}) \right], \ \ k = 0,1,\ldots, \tag{17}$$

where $G^k : V \times V \to \mathbb{R}$ are some given bilinear forms, $\gamma^k \in \mathbb{R}$ are iterative parameters, and $\mathbf{u}^k \in V$ is the k-th approximation to the exact solution of problem (16).

Theorem 3. [5] *Suppose that functional Φ satisfies the following properties:* $(\forall \mathbf{u} \in V)(\exists R_\Phi > 0)(\forall \mathbf{v} \in V)\{\,|\Phi(\mathbf{u},\mathbf{v})| \le R_\Phi \|\mathbf{v}\|_V\}$, $(\exists D_\Phi > 0)\ (\forall \mathbf{u},\mathbf{v},\mathbf{w} \in V)$ $\{\,|\Phi(\mathbf{u}+\mathbf{w},\mathbf{v}) - \Phi(\mathbf{u},\mathbf{v})\,| \le D_\Phi \|\mathbf{v}\|_V \|\mathbf{w}\|_V\}$, $(\exists B_\Phi > 0)(\forall \mathbf{u},\mathbf{v} \in V)\{\,\Phi(\mathbf{u}+\mathbf{v},\mathbf{v}) - \Phi(\mathbf{u},\mathbf{v}) \ge B_\Phi \|\mathbf{v}\|_V^2\}$. *Then nonlinear variational equation (16) has a unique solution $\bar{\mathbf{u}} \in V$. In addition, suppose that bilinear forms G^k, $k = 0,1,\ldots$ are symmetric, continuous with constant $M_G^* > 0$, coercive with constant $B_G^* > 0$, and the following conditions hold:* $(\exists k_0 \in \mathbb{N}_0)\,(\forall k \ge k_0)\ (\forall \mathbf{u} \in V)\ \{\,G^k(\mathbf{u},\mathbf{u}) \ge G^{k+1}(\mathbf{u},\mathbf{u})\}$, $(\exists \varepsilon \in (0,\gamma^*),\ \gamma^* = B_\Phi B_G^*/D_\Phi^2)(\exists k_1)(\forall k \ge k_1)\{\,\gamma^k \in [\varepsilon, 2\gamma^* - \varepsilon]\}$.
Then $\|\mathbf{u}^k - \bar{\mathbf{u}}\|_V \underset{k\to\infty}{\to} 0$, *where* $\{\mathbf{u}^k\} \subset V$ *is obtained by iterative method (17).*

5 Domain decomposition schemes

Now let us apply nonstationary iterative method (17) for solving the nonlinear variational equation (15), which corresponds to original contact problem. This equation can be written in form (16), where $\Phi(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) + J'(\mathbf{u},\mathbf{v})$, $Y(\mathbf{v}) = L(\mathbf{v})$, $\mathbf{u},\mathbf{v} \in V$, $V = V_0$, and iterative method (17) applied to solve (15) rewrites as follows:

$$G^k(\mathbf{u}^{k+1},\mathbf{v}) = G^k(\mathbf{u}^k,\mathbf{v}) - \gamma^k \left[A(\mathbf{u}^k,\mathbf{v}) + J'(\mathbf{u}^k,\mathbf{v}) - L(\mathbf{v}) \right], \ \ k = 0,1,\ldots. \tag{18}$$

Note, that in general case iterative method (18) does not lead to domain decomposition. Let us propose such variants of this method, which involve the domain decomposition. At first, let us take bilinear forms G^k in method (18) as follows:

$$G^k(\mathbf{u},\mathbf{v}) = \partial^2 F_1(\mathbf{u}^k,\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) + \partial^2 J(\mathbf{u}^k,\mathbf{u},\mathbf{v}), \ \ \mathbf{u},\mathbf{v} \in V_0, \tag{19}$$

$$\partial^2 J(\mathbf{u}^k,\mathbf{u},\mathbf{v}) = \sum_{\{\alpha,\beta\}\in Q} \int_{S_{\alpha\beta}} \chi_{\alpha\beta}^k \, g'_{\alpha\beta}(d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k) \left[u_{\alpha n} + u_{\beta n}\right] \left[v_{\alpha n} + v_{\beta n}\right] dS,$$

$$\chi_{\alpha\beta}^k = -[\operatorname{sgn}(d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)]^- = \{\,0,\ d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k \ge 0\,\} \vee \{\,1, \text{else}\,\}. \tag{20}$$

Here $\partial^2 F_1(\mathbf{u}^k,\mathbf{u},\mathbf{v})$, $\partial^2 J(\mathbf{u}^k,\mathbf{u},\mathbf{v})$ are one of the second subdifferentials of functionals F_1 and J in point $\mathbf{u}^k \in V_0$. In the case when $\gamma^k = 1$, $k = 0,1,\ldots$, iterative method (18) with bilinear forms (19) corresponds to semismooth Newton method for variational equation (15). However, this method does not lead to domain decomposition.

Now, let us take bilinear forms G^k in the following way:

$$G^k(\mathbf{u},\mathbf{v}) = A(\mathbf{u},\mathbf{v}) + X^k(\mathbf{u},\mathbf{v}), \;\; \mathbf{u},\mathbf{v} \in V_0, \tag{21}$$

$$X^k(\mathbf{u},\mathbf{v}) = \sum_{\alpha=1}^{N} \sum_{\beta\in B_\alpha} \int_{S_{\alpha\beta}} \psi^k_{\alpha\beta}\, g'_{\alpha\beta}(d_{\alpha\beta} - u^k_{\alpha n} - u^k_{\beta n})\, u_{\alpha n} v_{\alpha n}\, dS, \;\; \mathbf{u},\mathbf{v} \in V_0, \tag{22}$$

where $\psi^k_{\alpha\beta}(\mathbf{x}) = \{1, \mathbf{x} \in S^k_{\alpha\beta}\} \vee \{0, \mathbf{x} \in S_{\alpha\beta} \backslash S^k_{\alpha\beta}\}$ are characteristic functions of some given subsets $S^k_{\alpha\beta} \subseteq S_{\alpha\beta}$ of possible contact areas.

Iterative method (18) with bilinear forms (21) can be written in such way:

$$A(\tilde{\mathbf{u}}^{k+1},\mathbf{v}) + X^k(\tilde{\mathbf{u}}^{k+1},\mathbf{v}) = L(\mathbf{v}) + X^k(\mathbf{u}^k,\mathbf{v}) - J'(\mathbf{u}^k,\mathbf{v}), \;\; \forall \mathbf{v} \in V_0. \tag{23}$$

$$\mathbf{u}^{k+1} = \gamma^k \tilde{\mathbf{u}}^{k+1} + (1-\gamma^k)\,\mathbf{u}^k, \;\; k = 0,1,\ldots. \tag{24}$$

Since the common quantities of the subdomains are known from the previous iteration, variational equation (23) splits into N separate equations in subdomains Ω_α, and iterative method (23)–(24) can be written in the following equivalent form:

$$\begin{aligned} a_\alpha(\tilde{\mathbf{u}}^{k+1}_\alpha,\mathbf{v}_\alpha) &+ \sum_{\beta\in B_\alpha} \int_{S_{\alpha\beta}} \psi^k_{\alpha\beta}\, g'_{\alpha\beta}(d_{\alpha\beta} - u^k_{\alpha n} - u^k_{\beta n})\, \tilde{u}^{k+1}_{\alpha n} v_{\alpha n}\, dS = \\ &= l_\alpha(\mathbf{v}_\alpha) + \sum_{\beta\in B_\alpha} \int_{S_{\alpha\beta}} \psi^k_{\alpha\beta}\, g'_{\alpha\beta}(d_{\alpha\beta} - u^k_{\alpha n} - u^k_{\beta n})\, u^k_{\alpha n} v_{\alpha n}\, dS + \\ &+ \sum_{\beta\in B_\alpha} \int_{S_{\alpha\beta}} g^-_{\alpha\beta}(d_{\alpha\beta} - u^k_{\alpha n} - u^k_{\beta n})\, v_{\alpha n}\, dS, \;\; \forall \mathbf{v}_\alpha \in V^0_\alpha, \end{aligned} \tag{25}$$

$$\mathbf{u}^{k+1}_\alpha = \gamma^k \tilde{\mathbf{u}}^{k+1}_\alpha + (1-\gamma^k)\,\mathbf{u}^k_\alpha, \;\; \alpha = 1,2,\ldots,N, \;\; k = 0,1,\ldots. \tag{26}$$

In each iteration k of method (25)–(26), we have to solve N linear variational equations (25) in parallel, which correspond to linear elasticity problems in separate bodies Ω_α with Robin boundary conditions on possible contact areas. Therefore, this method refers to parallel Robin–Robin type domain decomposition schemes.

By taking different characteristic functions $\psi^k_{\alpha\beta}$, we can obtain different particular cases of domain decomposition method (25)–(26). Thus, taking $\psi^k_{\alpha\beta}(\mathbf{x}) \equiv 0$ ($S^k_{\alpha\beta} = \emptyset$), $\forall \alpha,\beta$, $\forall k$, we get parallel Neumann–Neumann domain decomposition scheme. Other borderline case is when $\psi^k_{\alpha\beta}(\mathbf{x}) \equiv 1$ ($S^k_{\alpha\beta} = S_{\alpha\beta}$), $\forall \alpha,\beta$, $\forall k$.

Moreover, we can choose characteristic functions $\psi^k_{\alpha\beta}$ by formula (20), i.e. $\psi^k_{\alpha\beta} = \chi^k_{\alpha\beta}$. Numerical experiments, provided by us, have shown, that such DDM has higher convergence rate than other particular domain decomposition schemes.

6 Numerical analysis

Numerical analysis of proposed DDMs has been provided for plane problem of unilateral contact between two isotropic bodies Ω_1 and Ω_2, one of which has a groove (Fig.1b). The bodies are loaded by normal stress with intensity $q = 10\,\text{MPa}$. Each body has length $l = 4\,\text{cm}$ and height $h = 1\,\text{cm}$. The elasticity constants of the bodies are the same: $E_1 = E_2 = 2.1 \cdot 10^5\,\text{MPa}$, $\nu_1 = \nu_2 = 0.3$. The distance between bodies is $d_{12}(\mathbf{x}) = r\left\{\left[1-(x_1-l)^2/b^2\right]^+\right\}^{3/2}$, $\mathbf{x} \in S_{12}$, where $b = 1\,\text{cm}$, $r = 5 \cdot 10^{-4}\,\text{cm}$.

Across possible contact area S_{12} there is a nonlinear Winkler layer. The relationship between normal contact stresses and displacements of this layer is described by the following power function: $g_{12}\left(w_{12}(\mathbf{x})\right) = B^{-1/a}\,\text{sgn}\left(w_{12}(\mathbf{x})\right)\left|w_{12}(\mathbf{x})\right|^{1/a}$, $\mathbf{x} \in S_{12}$, where parameters B and a are taken from the intervals $B \in [10^{-6}\,\text{cm}/(\text{MPa})^a,\ 2 \cdot 10^{-4}\,\text{cm}/(\text{MPa})^a]$, $a \in [0.1,\,1]$. For such choice of these parameters the nonlinear Winkler layer models a roughness of the possible contact surface [6].

This problem has been solved by DDM (25)–(26) with stationary iterative parameters $\gamma^k = \gamma$, $\forall k$ and characteristic functions ψ_{12}^k, taken by formula (20), i.e. $\psi_{12}^k = \chi_{12}^k$, $\forall k$. For solving linear variational problems (25) in each iteration k we have used finite element method with 8192 linear triangular elements for each body.

We have used the following initial guesses for displacements $u_{1n}^0(\mathbf{x}) = u_{2n}^0(\mathbf{x}) \equiv 10^{-4}\,\text{cm}$, and the next stopping criterion: $\rho_\alpha^{k+1} = \left\|u_{\alpha n}^{k+1} - u_{\alpha n}^k\right\|_2 / \left\|u_{\alpha n}^{k+1}\right\|_2 \le \varepsilon_u$, $\alpha = 1,2$, where $\|u_{\alpha n}\|_2 = \sqrt{\sum_j \left[u_{\alpha n}(\mathbf{x}^j)\right]^2}$ is discrete norm, $\mathbf{x}^j \in S_{12}$ are finite element nodes on the possible contact area, and $\varepsilon_u > 0$ is relative accuracy.

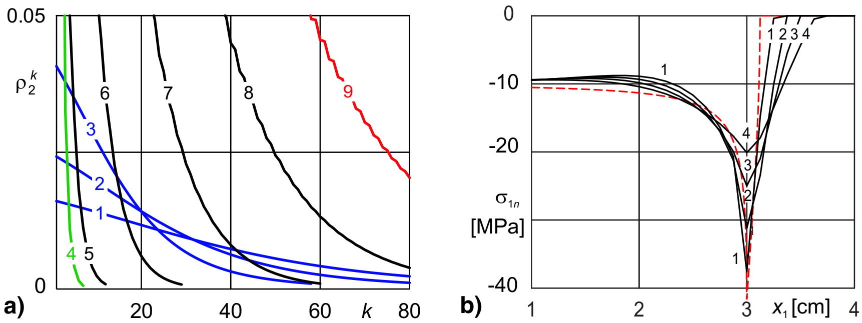

Fig. 2 Relative error (a), and normal contact stress (b)

At Fig.2a the relative error ρ_2^k of displacement u_{2n} on different iterations k, obtained for $B = 2.5 \cdot 10^{-5}\,\text{cm}/(\text{MPa})^a$, $a = 0.5$, is represented for different values of parameter γ. Curves 1–9 correspond to $\gamma = 0.02$, 0.03, 0.05, 0.6, 0.8 (0.3), 0.9, 0.95, 0.97, 0.98. For these values of parameter γ, DDM (25)–(26) reaches the accuracy $\varepsilon_u = 10^{-3}$ in 110, 83, 58, 7, 12 (14), 29, 60, 102, 155 iterations respectively. Thus, we conclude, that the best convergence rate reaches if $\gamma = 0.6$. The convergence rate is good if $\gamma \in [0.1,\,0.9]$. However, it becomes slow when γ is close to 0 or to 1. For $\gamma = 0.98$ the method is still convergent, but the convergence becomes nonmonotone.

We also have established, that the convergence rate of proposed DDMs does not deppend strongly on the number of finite element nodes m in each body. For $m = 43$, 149, 553, 2129, 8353, and 33089, DDM (25)–(26) with parameter $\gamma = 0.6$ reaches the accuracy $\varepsilon_u = 10^{-6}$ in 15, 15, 14, 14, 14, and 14 iterations respectively.

At Fig.2b the normal contact stress $\sigma_{1n} = \sigma_{2n}$, obtained by DDM (25)–(26) for $B = 10^{-5}\,\mathrm{cm}/(\mathrm{MPa})^a$ and different values of parameters a is represented. Curves 1–4 correspond to numerical solution for $a = 0.3, 0.6, 0.8, 1$. Dashed curve represents the analytical solution, obtained in [13] for contact between two halfspaces without nonlinear layer. Here we conclude, that for small values of a ($a \leq 0.3$) the influence of nonlinear layer on the contact behavior is not so large and the numerical solutions are close to the solution without layer. However, for larger values of a ($a \geq 0.5$) the influence of nonlinear layer becomes more significant and can not be neglected.

References

1. Bayada, G., Sabil, J., Sassi, T.: A Neumann–Neumann domain decomposition algorithm for the Signorini problem. Applied Mathematics Letters **17**(10), 1153–1159 (2004)
2. Bresch, D., Koko, J.: An optimization-based domain decomposition method for nonlinear wall laws in coupled systems. Math. Models Methods Appl. Sci. **14**(7), 1085–1101 (2004)
3. Dostál, Z., Kozubek, T., Vondrák, V., Brzobohatý, T., Markopoulos, A.: Scalable TFETI algorithm for the solution of multibody contact problems of elasticity. Int. J. Numer. Methods Engrg. **41**, 675–696 (2010)
4. Dyyak, I.I., Prokopyshyn, I.I.: Domain decomposition schemes for frictionless multibody contact problems of elasticity. In: G.K. et al. (ed.) Numerical Mathematics and Advanced Applications 2009, pp. 297–305. Springer (2010)
5. Dyyak, I.I., Prokopyshyn, I.I., Prokopyshyn, I.A.: Penalty Robin–Robin domain decomposition methods for unilateral multibody contact problems of elasticity: Convergence results (2012). URL `http://arxiv.org/pdf/1208.6478.pdf`
6. Goryacheva, I.G.: Contact mechanics in tribology. Kluwer (1998)
7. Hintermüller, M., Ito, K., Kunisch, K.: The primal-dual active set strategy as semismooth Newton method. SIAM J. OPTIM. **13**(3), 865–888 (2003)
8. Koko, J.: Convergence analysis of optimization-based domain decomposition methods for a bonded structure. Applied Numerical Mathematics (58), 69–87 (2008)
9. Koko, J.: Uzawa block relaxation domain decomposition method for a two-body frictionless contact problem. Applied Mathematics Letters **22**, 1534–1538 (2009)
10. Prokopyshyn, I.I.: Parallel domain decomposition schemes for frictionless contact problems of elasticity. Visnyk Lviv Univ. Ser. Appl. Math. Comp. Sci. **14**, 123–133 (2008). [In Ukrainian]
11. Prokopyshyn, I.I., Dyyak, I.I., Martynyak, R.M., Prokopyshyn, I.A.: Penalty Robin–Robin domain decomposition schemes for contact problems of nonlinear elastic bodies (2012). URL `http://arxiv.org/pdf/1209.1129.pdf`. [Accepted to DD20 Proceedings]
12. Sassi, T., Ipopa, M., Roux, F.X.: Generalization of Lion's nonoverlapping domain decomposition method for contact problems. Lect. Notes Comput. Sci. Eng. **60**, 623–630 (2008)
13. Shvets, R.M., Martynyak, R.M., Kryshtafovych, A.A.: Discontinuous contact of an anisotropic half-plane and a rigid base with disturbed surface. Int. J. Engng. Sci. **34**(2), 183–200 (1996)
14. Suquet, P.M.: Discontinuities and plasticity. In: CISM Courses Lect., 302, pp. 279–340 (1988)
15. Vorovich, I.I., Alexandrov, V.M. (eds.): Contact Mechanics. Fizmatlit, Moscow (2001)
16. Wohlmuth, B.: Variationally consistent discretization schemes and numerical algorithms for contact problems. Acta Numerica **20**, 569–734 (2011)