FETI-DP methods for Optimal Control Problems

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1 Introduction

We consider FETI-DP domain decomposition methods for optimal control problems of the form

$$\min_{y,u} \frac{1}{2} \int_\Omega (y(x) - y_d(x))^2 \, dx + \frac{\alpha}{2} \int_\Omega (u(x))^2 \, dx, \tag{1}$$

where $y \in V$ denotes the unknown state and $u \in U$ the unknown control, subject to a PDE constraint

$$a(y,v) = (f,v)_0 + (u,v)_0 \qquad \text{for all } v \in V. \tag{2}$$

The function y_d denotes a given desired state and $\alpha > 0$ a cost parameters. By $(\cdot,\cdot)_0$, we denote the standard L_2 inner product. In this paper, $a(\cdot,\cdot)$ will be the bilinear form associated with linear elasticity, i.e.,

$$a(y,v) = (2\mu\varepsilon(y), \varepsilon(v))_0 \; + \; (\lambda \operatorname{div} y, \operatorname{div} v)_0, \tag{3}$$

where μ, and λ are the Lamé parameters.

The state (displacement field) is sought in $V = H_0^1(\Omega, \partial\Omega_D)^2 = \{y \in H^1(\Omega)^2 : y = 0 \text{ on } \partial\Omega_D\}$, where $\Omega \subset \mathbf{R}^2$ and $\partial\Omega_D$ is part of its boundary. For simplicity, we consider the case of volume control, i.e., $U = L_2(\Omega)^2$.

Dual-primal FETI methods were first introduced by Farhat, Lesoinne, Le Tallec, Pierson, and Rixen [3] and have successfully scaled to 10^5 processor cores [6]. In [8] a first convergence bound for scalar problems in 2D was provided. Numerical scalability for FETI-DP methods applied to linear elasticity problems was first proven in [7].

Balancing Neumann-Neumann domain decomposition methods for the optimal control of scalar problems have been considered in Heinkenschloss and Nguyen [5, 4]. There, local optimal control problems on non-overlapping subdomains are considered and a Balancing Neumann-Neumann preconditioner is constructed for the indefinite Schur complement. Multigrid methods have, of course, also been considered for optimal control problems, see, e.g., [10]. A review of block approaches to optimal control problems can be found in [9]. A recent block approach can be found in [11].

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We discretize y by $P1$ finite elements, u by $P0$ finite elements and obtain the discrete problem

$$\min_{y,u} \frac{1}{2} y^T M y + \frac{\alpha}{2} u^T Q u - c^T y \tag{4}$$

$$\text{s.t.} \qquad Ay = f + Nu. \tag{5}$$

2 Discrete Problem and Domain Decomposition

The necessary and sufficient optimality conditions are given by the discrete system

$$\begin{bmatrix} M & 0 & A^T \\ 0 & \alpha Q & -N^T \\ A & -N & 0 \end{bmatrix} \begin{bmatrix} y \\ u \\ p \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ f \end{bmatrix} \tag{6}$$

where $A \in \mathbf{R}^{n\times n}$, $Q \in \mathbf{R}^{m\times m}$, $M \in \mathbf{R}^{n\times n}$. Here, $A = A^T = (a(\varphi_i, \varphi_j))_{i,j}$ is a stiffness matrix, whereas $Q = (\langle \psi_i, \psi_j \rangle)_{i,j}$, $M = (\langle \psi_i, \psi_j \rangle)_{i,j}$ and $N = (\langle \varphi_i, \psi_j \rangle)_{i,j}$ are mass matrices. We will denote the block system (6) by

$$Kx = b. \tag{7}$$

We decompose Ω into N nonoverlapping subdomains $\Omega_i, i = 1,\ldots,N$, i.e. $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega_i}$, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Each subdomain is the union of shape-regular finite element cells with matching nodes across the interface, $\Gamma := \bigcup_{i\neq j} \partial\Omega_i \cap \partial\Omega_j$, where $\partial\Omega_i, \partial\Omega_j$ are the boundaries of Ω_i, Ω_j, respectively.

For each subdomain, we assemble the local problem $K^{(i)}$, which represents the discrete optimality system for (1)–(2), restricted to the subdomain Ω_i. Let us denote, for each subdomain, the variables that are on the subdomain interface by an index Γ and the interior unknowns by I. Note that the interior variables also comprise the variables on the Neumann boundary $\partial\Omega \setminus \partial\Omega_D$. In block form, we can now write the subdomain problem matrices $K^{(i)}, i = 1,\ldots,N$ as

$$K^{(i)} = \begin{bmatrix} M^{(i)} & 0 & A^{(i)T} \\ 0 & \alpha Q^{(i)} & -N^{(i)T} \\ A^{(i)} & -N^{(i)} & 0 \end{bmatrix} = \left[\begin{array}{cc|c|cc} M_{II}^{(i)} & M_{I\Gamma}^{(i)} & 0 & A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ M_{I\Gamma}^{(i)T} & M_{\Gamma\Gamma}^{(i)} & 0 & A_{I\Gamma}^{(i)T} & A_{\Gamma\Gamma}^{(i)} \\ \hline 0 & 0 & \alpha Q_{II}^{(i)} & -N_{II}^{(i)T} & -N_{\Gamma I}^{(i)T} \\ \hline A_{II}^{(i)} & A_{I\Gamma}^{(i)} & -N_{II}^{(i)} & 0 & 0 \\ A_{I\Gamma}^{(i)T} & A_{\Gamma\Gamma}^{(i)} & -N_{\Gamma I}^{(i)} & 0 & 0 \end{array}\right]. \tag{8}$$

We define the block matrices

$$K_{II}^{(i)} = \begin{bmatrix} M_{II}^{(i)} & 0 & A_{II}^{(i)} \\ 0 & \alpha Q_{II}^{(i)} & -N_{II}^{(i)} \\ A_{II}^{(i)} & -N_{II}^{(i)} & 0 \end{bmatrix}, \; K_{\Gamma\Gamma}^{(i)} = \begin{bmatrix} M_{\Gamma\Gamma}^{(i)} & A_{\Gamma\Gamma}^{(i)} \\ A_{\Gamma\Gamma}^{(i)} & 0 \end{bmatrix}, \; K_{I\Gamma}^{(i)} = \begin{bmatrix} M_{I\Gamma}^{(i)} & A_{I\Gamma}^{(i)} \\ 0 & -N_{\Gamma I}^{(i)T} \\ A_{I\Gamma}^{(i)} & 0 \end{bmatrix}. \tag{9}$$

Following the approach of FETI-type methods a continuity constraint $Bx = 0$ is introduced to enforce the continuity of y and p across each interface Γ. The introduction of Lagrange multipliers λ then leads to the FETI master system

$$\begin{bmatrix} K^{(1)} & & & \widehat{B}^{(1)} \\ & \ddots & & \vdots \\ & & K^{(N)} & \widehat{B}^{(N)} \\ \widehat{B}^{(1)} & \dots & \widehat{B}^{(N)} & 0 \end{bmatrix} \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(N)} \\ \lambda \end{bmatrix} = \begin{bmatrix} b^{(1)} \\ \vdots \\ b^{(N)} \\ 0 \end{bmatrix}. \tag{10}$$

In the context of our optimal control problem, $\widehat{B}^{(i)}$ is of the form $\widehat{B}^{(i)} = \left[B_y^{(i)} \middle| 0 \middle| B_p^{(i)} \right]$. Note that it is not appropriate to enfore continuity for the control variable u, since it is an algebraic variable and has been discretized by discontinuous elements.

In dual-primal FETI methods the continuity constraint is enforced on a subset of the variables on the interface Γ by partial finite element assembly. These variables are denoted by the index Π (primal). Here, for our 2D problems, we use primal vertex variables. For the remaining interface variables, the continuity is enforced by Lagrange multipliers. Such interface variables are denoted by the index Δ (dual). We thus write the matrices $M^{(i)}, A^{(i)}, N^{(i)}$ appearing in (8) in the form

$$M^{(i)} = \begin{bmatrix} M_{II}^{(i)} & M_{I\Delta}^{(i)} & M_{I\Pi}^{(i)} \\ M_{I\Delta}^{(i)T} & M_{\Delta\Delta}^{(i)} & M_{\Delta\Pi}^{(i)} \\ M_{I\Pi}^{(i)T} & M_{I\Delta}^{(i)T} & M_{\Pi\Pi}^{(i)} \end{bmatrix}, A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} & A_{I\Pi}^{(i)} \\ A_{I\Delta}^{(i)T} & A_{\Delta\Delta}^{(i)} & A_{\Delta\Pi}^{(i)} \\ A_{I\Pi}^{(i)T} & A_{I\Delta}^{(i)T} & A_{\Pi\Pi}^{(i)} \end{bmatrix}, N^{(i)} = \begin{bmatrix} N_{II}^{(i)} \\ N_{\Delta I}^{(i)} \\ N_{\Pi I}^{(i)} \end{bmatrix}, \tag{11}$$

and $Q^{(i)} = Q_{II}^{(i)}$. Inserting this block form into (8), we obtain the block form of $K_{\Pi\Pi}^{(i)}$,

$$K_{\Pi\Pi}^{(i)} = \begin{bmatrix} M_{\Pi\Pi}^{(i)} & A_{\Pi\Pi}^{(i)T} \\ A_{\Pi\Pi}^{(i)} & 0 \end{bmatrix}. \tag{12}$$

For the assembly of the primal variables y_Π and p_Π, we define the combined assembly operator $\widehat{R}_{\Pi}^{(i)T}$, i.e., we obtain for the assembled global matrix $\widetilde{K}_{\Pi\Pi}$

$$\begin{aligned} \widetilde{K}_{\Pi\Pi} &= \widehat{R}_{\Pi}^{T} K_{\Pi\Pi} \widehat{R}_{\Pi} = \left[\widehat{R}_{\Pi}^{(1)T}, \dots, \widehat{R}_{\Pi}^{(N)T} \right] \begin{bmatrix} K_{\Pi\Pi}^{(1)} & & 0 \\ & \ddots & \\ 0 & & K_{\Pi\Pi}^{(N)} \end{bmatrix} \begin{bmatrix} \widehat{R}_{\Pi}^{(1)} \\ \\ \widehat{R}_{\Pi}^{(N)} \end{bmatrix} \\ &= \sum_{i=1}^{N} \widehat{R}_{\Pi}^{(i)T} K_{\Pi\Pi}^{(i)} \widehat{R}_{\Pi}^{(i)} = \sum_{i=1}^{N} \begin{bmatrix} R_{\Pi}^{(i)T} & 0 \\ 0 & R_{\Pi}^{(i)T} \end{bmatrix} \begin{bmatrix} M_{\Pi\Pi}^{(i)} & A_{\Pi\Pi}^{(i)T} \\ A_{\Pi\Pi}^{(i)} & 0 \end{bmatrix} \begin{bmatrix} R_{\Pi}^{(i)} & 0 \\ 0 & R_{\Pi}^{(i)} \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{M}_{\Pi\Pi} & \widetilde{A}_{\Pi\Pi}^{T} \\ \widetilde{A}_{\Pi\Pi} & 0 \end{bmatrix}. \end{aligned} \tag{13}$$

The partially assembled system matrix is then

$$\widetilde{K} = \begin{bmatrix} K_{BB}^{(1)} & & & \widetilde{K}_{B\Pi}^{(1)} \\ & \ddots & & \vdots \\ & & K_{BB}^{(N)} & \widetilde{K}_{B\Pi}^{(N)} \\ \widetilde{K}_{B\Pi}^{(1)T} & \dots & \widetilde{K}_{B\Pi}^{(N)T} & \widetilde{K}_{\Pi\Pi} \end{bmatrix} \tag{14}$$

with the blocks

$$K_{BB}^{(i)} = \begin{bmatrix} M_{II}^{(i)} & M_{I\Delta}^{(i)} & 0 & A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ M_{I\Delta}^{(i)T} & M_{\Delta\Delta}^{(i)} & 0 & A_{I\Delta}^{(i)T} & A_{\Delta\Delta}^{(i)} \\ 0 & 0 & \alpha Q_{II}^{(i)} & -N_{II}^{(i)T} & -N_{\Delta I}^{(i)T} \\ A_{II}^{(i)} & A_{I\Delta}^{(i)} & -N_{II}^{(i)} & 0 & 0 \\ A_{I\Delta}^{(i)T} & A_{\Delta\Delta}^{(i)} & -N_{\Delta I}^{(i)} & 0 & 0 \end{bmatrix}, \tag{15}$$

and

$$\begin{aligned} \widetilde{K}_{B\Pi}^{(i)T} &= \begin{bmatrix} \widetilde{M}_{I\Pi}^{(i)T} & \widetilde{M}_{\Delta\Pi}^{(i)T} & 0 & \widetilde{A}_{I\Pi}^{(i)T} & \widetilde{A}_{\Delta\Pi}^{(i)T} \\ \widetilde{A}_{I\Pi}^{(i)T} & \widetilde{A}_{\Delta\Pi}^{(i)T} & \widetilde{N}_{I\Pi}^{(i)T} & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R_{\Pi}^{(i)T} & 0 \\ 0 & R_{\Pi}^{(i)T} \end{bmatrix} \begin{bmatrix} \widetilde{M}_{I\Pi}^{(i)T} & \widetilde{M}_{\Delta\Pi}^{(i)T} & 0 & \widetilde{A}_{I\Pi}^{(i)T} & \widetilde{A}_{\Delta\Pi}^{(i)T} \\ \widetilde{A}_{I\Pi}^{(i)T} & \widetilde{A}_{\Delta\Pi}^{(i)T} & \widetilde{N}_{I\Pi}^{(i)T} & 0 & 0 \end{bmatrix}. \end{aligned} \tag{16}$$

Now, we can formulate the FETI-DP master system,

$$\begin{bmatrix} \widetilde{K} & \widehat{B}^T \\ \widehat{B} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tilde{b} \\ 0 \end{bmatrix}, \qquad u \in \mathbf{R}^n, \lambda \in \mathbf{R}^m, \tag{17}$$

from which the solution of the original finite element problem (6) can be obtained by averaging the solution $\tilde{x}$ from (17) in the interface variables. Here, the jump operator $\widehat{B}$ only acts on the variables y_Δ and p_Δ. The vectors $\tilde{x}$ and $\tilde{b}$ have the form

$$\begin{aligned} x^T &= \Big[[y_I^{(i)T}, y_\Delta^{(i)T}, u_I^{(i)T}, p_I^{(i)T}, p_\Delta^{(i)T}], \dots, [y_I^{(N)T}, y_\Delta^{(N)T}, u_I^{(N)T}, p_I^{(N)T}, p_\Delta^{(N)T}], [\tilde{y}_\Pi^T, \tilde{p}_\Pi^T]\Big] \\ b^T &= \Big[[c_I^{(i)T}, c_\Delta^{(i)T}, 0, f_I^{(i)T}, f_\Delta^{(i)T}], \dots, [c_I^{(N)T}, c_\Delta^{(N)T}, 0, f_I^{(N)T}, f_\Delta^{(N)T}], [\tilde{c}_\Pi^T, \tilde{f}_\Pi^T]\Big] \end{aligned}$$

After the elimination of x in (17) it remains to solve a system

$$F\lambda = d \tag{18}$$

where F is symmetric indefinite, i.e., with positive and negative eigenvalues, by a suitable Krylov subspace method. The FETI-DP coarse problem is

$$\widetilde{S}_{\Pi\Pi} = \widetilde{K}_{\Pi\Pi} - \sum_{i=1}^{N} \widetilde{K}_{B\Pi}^{(i)} \widetilde{K}_{\Pi\Pi}^{(i)} \widetilde{K}_{B\Pi}^{(i)T}. \tag{19}$$

To define the Dirichlet preconditioner, we consider the block submatrices of $K^{(i)}$ defined in (9),

$$K^{(i)} = \begin{bmatrix} K_{II}^{(i)} & K_{\Gamma I}^{(i)\,T} \\ K_{\Gamma I}^{(i)} & K_{\Gamma\Gamma}^{(i)} \end{bmatrix}. \tag{20}$$

Let us define the Schur complement

$$S_{\Gamma\Gamma} = \sum_{i=1}^{N} (K_{\Gamma\Gamma}^{(i)} - K_{\Gamma I}^{(i)} (K_{II}^{(i)})^{-1} K_{\Gamma I}^{(i)T}) = \sum_{i=1}^{N} S_{\Gamma\Gamma}^{(i)}, \tag{21}$$

which can be computed completely in parallel. The Dirichlet preconditioner is then given in matrix form by

$$M^{-1} = B_D \widehat{R}_\Gamma^T S_{\Gamma\Gamma} \widehat{R}_\Gamma B_D^T = \sum_{i=1}^{N} B_D^{(i)} \widehat{R}_\Gamma^{(i)T} S_{\Gamma\Gamma}^{(i)} \widehat{R}_\Gamma^{(i)} B_D^{(i)T}, \tag{22}$$

where B_D is a variant of the jump operator B scaled by the inverse multiplicity of the node. The matrices $R_\Gamma^{(i)}$ are simple restriction operators which restrict the nonprimal degrees of freedom of a subdomain to the interface, i.e. $\widehat{R}_\Gamma^{(i)} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, if the variables are numbered $[I, \Delta]$ on the right hand side and $[\Delta, \Pi]$ on the left hand side of the operator.

3 Well-posedness of the local problems

In [4] the well-posedness of the local subdomain problems for the balancing Neumann-Neumann method was considered. These considerations are also valid for FETI-1-type methods. In contrast to FETI-1 and Balancing Neumann-Neumann methods the coarse problems of the more recent FETI-DP and BDDC methods are constructed from partial finite element assembly.

We therefore briefly comment on the well-posedness of the subdomain problems, i.e. the local blocks $K_{BB}^{(i)}$ in (14), as well as the coarse problem (19). Each block $K_{BB}^{(i)}$ represents a discrete optimality system local to the subdomain Ω_i. In contrast to the original problem (2), natural (stress) boundary conditions are imposed on $\partial\Omega_i$ for the state y, except in the (few) primal degrees of freedom on the interface boundary, and except for the degrees of freedom on $\partial\Omega_i \cap \partial\Omega_D$, where Dirichlet conditions apply. These conditions are sufficient to exclude rigid body motions. Consequently, the local elasticity system (the four A blocks in (15) combined), is well posed, and thus it is straightforward to show that also the optimality system is well posed, whence $K_{BB}^{(i)}$ non-singular. The non-singularity of the total matrix $\widetilde{K}$ in (14) can be shown along the same lines. And thus the non-singularity of the Schur complement (19) follows.

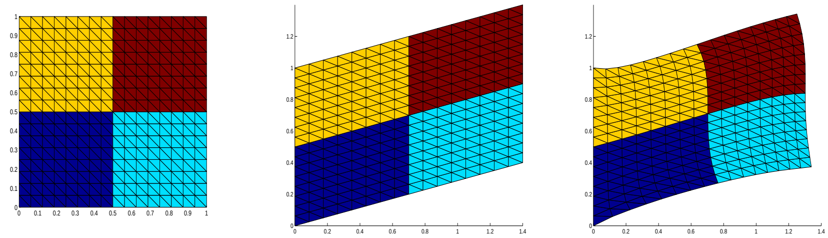

Fig. 1 Model problem: Undeformed configuration, desired state, and solution computed using FETI-DP.

Finally, (21) is well defined since $K_{II}^{(i)}$ is non-singular. Note that each $K_{II}^{(i)}$ represents a discrete optimality system with all-Dirichlet boundary conditions on $\partial\Omega_i$ for the state and adjoint states, with these boundary degrees of freedom removed.

4 Numerical Results

Here we will report on the use of GMRES applied to the symmetric indefinite FETI-DP system (18), using the symmetric indefinite Dirichlet preconditioner (22). Note that there is no theory for the convergence of GMRES in this situation. The numerical results are nevertheless very encouraging. We also report on the convergence of QMR. The stopping criterion is the relative reduction of the preconditioned residual by 10 orders of magnitude. In [5, 4] a symmetric QMR was used for the Neumann-Neumann method. The numerical results are nevertheless very encouraging. The iteration counts using QMR and GMRES are very similar.

We consider the volume control of a linear elastic problem on the unit square. The desired displacement y_d is a obtained from applying a linear transformation to the unit square, i.e., $y_d(x,y) = (\frac{2}{5}x, \frac{2}{5}y)^T$; see Fig. 1. The Dirichlet boundary is on the left. The material data is $E = 1$ (Young's modulus) and $\nu = 0.3$ (Poisson's ratio) in all cases, which are related to the Lamé constants via $E = \frac{\mu\,(2\mu+3\lambda)}{\mu+\lambda}$ and $\nu = \frac{\lambda}{2\,(\mu+\lambda)}$.

We numerically observe scalability with respect to the number of subdomains as known for CG in the symmetric positive case, i.e., the number of iterations approaches a limit for an increasing number of subdomains N if H/h is maintained fixed, see Tab. 1. Moreover the number of iterations grows only weakly with H/h for a fixed number of subdomains N, see Tab. 2. In Tab. 3 we see that the methods shows robustness with respect to α. In Tab. 4 we report on the strong parallel scalability of the largest problem from Tab. 2 using the GMRES implementation from PETSc [1]. We have used UMFPACK 4.3 [2] for the solution of the subdomain problems.

DIRICHLET PRECONDITIONER - Weak Scaling - GMRES and QMR

N	#Points	#Elem	#gmres	#qmr	#Points	#Elem	#gmres	#qmr	#Points	#Elem	#gmres	#qmr
	$H/h=2$				$H/h=4$				$H/h=8$			
2×2	25	32	**8**	**9**	81	128	**11**	**11**	289	512	**13**	**14**
4×4	81	128	**14**	**14**	289	512	**19**	**20**	1 089	2 048	**25**	**27**
6×6	169	288	**15**	**16**	625	1 152	**22**	**24**	2 401	4 608	**30**	**32**
8×8	289	512	**15**	**16**	1 089	2 048	**24**	**25**	4 225	8 192	**32**	**34**
10×10	441	800	**16**	**16**	1 681	3 200	**24**	**25**	6 561	12 800	**33**	**36**
12×12	625	1 152	**16**	**17**	2 401	4 608	**25**	**26**	9 409	18 432	**34**	**38**
16×16	1 089	2 048	**16**	**17**	4 225	8 192	**25**	**26**	16 641	32 768	**35**	**38**
20×20	1 681	3 200	**16**	**17**	6 561	12 800	**25**	**26**	25 921	51 200	**36**	**39**
24×24	2 401	4 608	**16**	**18**	9 409	18 432	**25**	**26**	37 249	73 728	**36**	**39**
28×28	3 249	6 272	**16**	**18**	12 769	25 088	**26**	**26**	50 625	100 352	**36**	**40**
32×32	4 225	8 192	**16**	**18**	16 641	32 768	**26**	**27**	66 049	131 072	**37**	**40**
36×36	5 329	10 368	**16**	**18**	21 025	41 472	**26**	**27**	83 521	165 888	**37**	**41**
40×40	6 561	12 800	**16**	**18**	25 921	51 200	**26**	**27**	103 041	204 800	**37**	**41**
48×48	9 409	18 432	**16**	**18**	37 249	73 728	**26**	**27**	148 225	294 912	**37**	**41**
56×56	12 769	25 088	**16**	**18**	50 625	100 352	**26**	**27**	201 601	401 408	**37**	**41**
64×64	16 641	32 768	**16**	**19**	66 049	131 072	**26**	**27**	263 169	524 288	**37**	**41**

Table 1 Weak scaling. The number of GMRES and QMR iterations is scalable with respect to the number of subdomains, i.e., it is bounded independently of N. $\alpha = 0.01$. Material parameters $E = 1$, $\nu = 0.3$. The iteration is stopped when the preconditioned residual has been reduced by 10 orders of magnitudes. The largest problem has $2\,101\,252 = 4\times 263\,169 + 2\times 524\,288$ d.o.f.

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DIRICHLET PRECONDITIONER - GMRES and QMR

H/h	#Points	#Elem	#gmres	#qmr	#Points	#Elem	#gmres	#qmr	#Points	#Elem	#gmres	#qmr
	$N = 2\times 2$				$N = 3\times 3$				$N = 4\times 4$			
2	25	32	**8**	**9**	49	72	**12**	**13**	81	128	**14**	**14**
4	81	128	**11**	**11**	169	288	**16**	**17**	289	512	**19**	**20**
6	169	288	**12**	**12**	361	648	**18**	**19**	625	1 152	**23**	**24**
8	289	512	**13**	**14**	625	1 152	**20**	**21**	1 089	2 048	**25**	**27**
12	625	1 152	**14**	**14**	1 369	2 592	**23**	**25**	2 401	4 608	**28**	**31**
16	1 089	2 048	**14**	**15**	2 401	4 609	**25**	**27**	4 225	8 192	**31**	**34**
24	2 401	4 608	**16**	**16**	5 329	10 368	**28**	**30**	9 409	18 432	**35**	**36**
32	4 225	8 192	**16**	**17**	9 409	18 432	**29**	**30**	16 641	32 768	**37**	**38**
48	9 409	18 432	**17**	**18**	21 025	41 472	**32**	**33**	37 249	73 728	**40**	**43**
64	16 641	32 768	**18**	**19**	37 249	73 728	**33**	**35**	66 049	131 072	**43**	**45**
96	37 249	73 728	**19**	**19**	83 521	165 888	**34**	**38**	148 225	294 912	**46**	**50**
128	66 049	131 072	**19**	**20**	148 225	294 912	**36**	**39**	263 169	524 288	**49**	**52**

Table 2 The number of GMRES and QMR iterations grows only weakly with the subdomain size. $\alpha = 0.01$. Material parameters $E = 1$, $\nu = 0.3$. The iteration is stopped when the preconditioned residual has been reduced by 10 orders of magnitudes. The largest problem has $2\,102\,452 = 2 \times 2 \times 263\,169 + 524\,288$ d.o.f.

DIRICHLET PRECONDITIONER
- GMRES and QMR

N	H/h	α	#gmres	#qmr
8×8	4	1	**19**	**20**
8×8	4	0.1	**22**	**22**
8×8	4	0.01	**24**	**25**
8×8	4	0.001	**23**	**24**
8×8	4	0.000 1	**19**	**21**

Table 3 Dependence on α. The preconditioner is robust with respect to the choice of the cost parameter $\alpha > 0$.

#Cores	N	H/h	#Points	#Elem	d.o.f.	#gmres	Time
1	4×4	64	66 049	131 072	526 340	49	$89.7s$
2	4×4	64	66 049	131 072	526 340	49	$45.6s$
4	4×4	64	66 049	131 072	526 340	49	$23.9s$
8	4×4	64	66 049	131 072	526 340	49	$14.2s$
16	4×4	64	66 049	131 072	526 340	49	$10.7s$

Table 4 Strong parallel scalability on a 16 core Opteron 8380 server (2.5 Ghz) for one of the problems from Tab. 2.

10. Schöberl, J., Simon, R., Zulehner, W.: A robust multigrid method for elliptic optimal control problems. SIAM J. Numer. Anal. **49**(4), 1482–1503 (2011)
11. Schöberl, J., Zulehner, W.: Symmetric indefinite preconditioners for saddle point problems with applications to PDE-constrained optimization problems. SIAM J. Matrix Anal. Appl. **29**(3), 752–773 (electronic) (2007)