A non overlapping domain decomposition method for the obstacle problem

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Abstract
We present a non-overlapping domain decomposition method for the obstacle problem. In this approach, the original problem is reformulated into two subproblems such that the first problem is a variational inequality in subdomain Ω^i and the other is a variational equality in the complementary subdomain Ω^e, where Ω^e and Ω^i are multiply-connected, in general. The main challenge is to obtain the global solution through coupling of the two subdomain solutions, which requires the solution of a nonlinear interface problem. This is achieved via a fixed point iteration. This new formulation reduces the computational cost as the variational inequality is solved in a smaller region. The algorithm requires some mild assumption about the location of Ω^i, which is the domain containing the region corresponding to the coincidence set. Numerical experiments are included to illustrate the performance of the resulting method.

Key words: non-overlapping, domain decomposition method, obstacle problem

1 Obstacle problem

The obstacle problem is to determine the equilibrium position of an elastic membrane in a domain $\Omega \subseteq \mathbb{R}^2$ with closed boundary $\partial\Omega$, which lies above an obstacle function $\psi : \Omega \to \mathbb{R}^+$ under the vertical force f. The classical solution u of this model problem is the vertical displacement of the membrane. Since the membrane is fixed on $\partial\Omega$, we have boundary conditions of Dirichlet type (say $u = 0$). The problem can be written as

$$\begin{cases} -\Delta u - f \geq 0 & \text{in } \Omega, \\ u - \psi \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

subject to the pointwise complementarity condition $(u-\psi)(-\Delta u - f) = 0$.
Let $\mathscr{C} = \{\mathbf{x} \in \Omega : u(\mathbf{x}) = \psi(\mathbf{x})\}$ denote the coincidence set. Then the complementarity conditions yields the PDE $-\Delta u - f = 0$ in $\Omega \setminus \mathscr{C}$. The weak formulation of (1) can be written as [10]

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$$\begin{cases} \text{Find } u \in K \text{ such that } \forall v \in K, \\ a(u, v-u) \geq (f, v-u), \end{cases} \tag{2}$$

which can be shown to be equivalent to the following minimization problem

$$\begin{cases} \text{Find } u \in K, \text{ such that } \forall\, v \in K, \\ J(u) \leq J(v), \end{cases}$$

where $K = \{v \in V := H_0^1(\Omega) : \ v \ \geq \ \psi \text{ in } \Omega\}$ is convex and

$$J(v) \ = \ \frac{1}{2}a(\nabla v, \nabla v) - (f, v), \ \ (f, v) = \int_\Omega f v d\Omega, \ \ a(u, v) = \int_\Omega \nabla u \cdot \nabla v d\Omega.$$

An important class of solution techniques for (2) is that of multilevel and multigrid methods for constrained minimization problems, first introduced by [6] and [2] some variants of these method were studied in [4] and were analyzed in [5]. A challenging task for multigrid is the representation of the coincidence set on a coarse grid, as shown in the review paper [3]. Some multi-grid and two level domain decomposition methods are given in [9] [7] in which it is shown that the overlapping DDM has a linear convergence for constrained obstacle problem if the obstacle and computed functions decomposed properly. Some more variants of multi-grid method are given in [1], where the decomposition of the closed convex set for minimization problem is introduced as a sum of closed convex level subsets; the convergence rate is shown to depend on the number of levels.

2 A non-overlapping domain decomposition method

Let Ω^i denote an open subset of Ω containing the coincidence set $\mathscr{C}$ and let $\Omega^e = \Omega \setminus \bar{\Omega}^i$. Let Γ denote the interface between Ω^i and Ω^e. This decomposition allows us to reformulate our problem into two subproblems: one which is a partial differential inequality (PDI) in subdomain Ω^i and the other which is a partial differential equation (PDE) in Ω^e.

Let $z = u|_{\Omega^e}, w = u|_{\Omega^i}$, $f^e = f|_{\Omega^e}$ and $f^i = f|_{\Omega^i}$ be the restrictions of u and f to Ω^e and Ω^i respectively; let also $\lambda = u|_\Gamma$ be the trace of u on Γ. Assuming for now that λ is known, problem (1) decouples into the two subproblems

$$\text{PDE:} \begin{cases} -\Delta z = f^e & \text{in } \Omega^e, \\ z = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ z = \lambda & \text{on } \Gamma, \end{cases} \qquad \text{PDI:} \begin{cases} -\Delta w \geq f^i & \text{in } \Omega^i, \\ w \geq \psi^i & \text{in } \Omega^i, \\ w = 0 & \text{on } \partial\Omega \setminus \Gamma, \\ w = \lambda & \text{on } \Gamma. \end{cases}$$

with $(-\Delta w - f^i)(w - \psi^i) = 0$ satisfied in a pointwise sense in Ω^i. The subproblem PDE can be further decoupled as follows:

$$PDE_1 : \begin{cases} -\Delta z_1 = f^e & \text{in } \Omega^e, \\ z_1 = \ 0 & \text{on } \partial\Omega \setminus \Gamma, \\ z_1 = 0 & \text{on } \Gamma, \end{cases} \quad PDE_2 : \begin{cases} -\Delta z_2 = 0 & \text{in } \Omega^e, \\ z_2 = \ 0 & \text{on } \partial\Omega \setminus \Gamma, \\ z_2 = \lambda & \text{on } \Gamma, \end{cases} \tag{3}$$

where $z|_{\Omega^e} = z_1 + z_2$ with $z_2 = E\lambda$ where E is the harmonic extension operator to Ω^e. Writing the weak formulation (2) as

$$a^e(z, v-z) + a^i(w, v-w) \geq (f^e, v-z)_{\Omega^e} + (f^i, v-w)_{\Omega^i}, \tag{4}$$

where

$$a^e(z,v) = \int_{\Omega^e} \nabla z \cdot \nabla v \, d\Omega^e \text{ and } a^i(w,v) = \int_{\Omega^i} \nabla w \cdot \nabla v \, d\Omega^i$$

the variational formulations of (3) and PDI are

$$\begin{cases} \text{find } z_1 \in H_0^1(\Omega^e) \text{ such that } \forall v \in H_0^1(\Omega^e) \\ a^e(z_1, v-z) - \displaystyle\int_\Gamma \mathbf{n}_1 \cdot \nabla z_1 \cdot (v-z) \, d\Gamma = (f^e, v-z)_{\Omega^e}, \end{cases} \tag{5}$$

$$\begin{cases} \text{find } z_2 \in H^1(\Omega^e) \text{ such that } \forall v \in H^1(\Omega^e) \\ a^e(z_2, v-z) - \displaystyle\int_\Gamma \mathbf{n}_1 \cdot \nabla z_2 \cdot (v-z) \, d\Gamma = 0, \end{cases} \tag{6}$$

$$\begin{cases} \text{find } w \in H^1(\Omega^i) \text{ such that } \forall v \in H^1(\Omega^i) \\ a^i(w, v-w) - \displaystyle\int_\Gamma \mathbf{n}_2 \cdot \nabla w \cdot (v-w) d\Gamma \geq (f^i, v-w)_{\Omega^i}. \end{cases} \tag{7}$$

For $i = 1,2$, $\mathbf{n}_i$, is the normal direction from Ω^e and Ω^i respectively. Adding the above weak formulations, where $z_1 = 0$, $z_2 = \lambda = w$ on Γ and using the weak formulation (4) yields a partial Steklov-Poincaré inequality for λ (corresponding to the splitting of PDE)

$$(\mathscr{S}^e \lambda, \mu - \lambda) \leq (g(\lambda), \mu - \lambda).$$

Using the assumption that the interface Γ lies outside the support of the obstacle we obtain the following nonlinear equation on the interface

$$(\mathscr{S}^e \lambda, \mu) = (g(\lambda), \mu). \tag{8}$$

The Steklov-Poincaré operator $\mathscr{S}^e : \Lambda \to \Lambda'$ (where $\Lambda = H^{1/2}(\Gamma)$, $H_0^{1/2}(\Gamma)$ or $H_{00}^{1/2}(\Gamma)$ depending on the nature of the problem) is defined as

$$(\mathscr{S}^e \lambda, \mu) := \int_\Gamma (\mathbf{n_1} \cdot \nabla(E\lambda)) \mu \, d\Gamma,$$

and

$$(g(\lambda), \mu) := -\int_\Gamma (\mathbf{n_1} \cdot \nabla z_1 + \mathbf{n_2} \cdot \nabla w) \mu \, d\Gamma$$

Applying Green's formula we get the alternative representation of $\mathscr{S}^e$

$$(\mathscr{S}^e \lambda, \mu) := a^e(E\lambda, F\mu) \quad \forall \lambda, \mu \in \Lambda$$

where F denotes an arbitrary extension operator to Ω^e. By using the above definition of $\mathscr{S}^e$, our classical problem can be written as an ordered sequence of three decoupled problems involving Poisson problem on subdomain Ω^e together with a problem set on the interface Γ which is coupled with the problem on Ω^i.

$$\begin{cases} -\Delta z_1 = f^e \text{ in } \Omega^e, \\ z_1 = 0 \text{ on } \partial\Omega \setminus \Gamma, \\ z_1 = 0 \text{ on } \Gamma, \end{cases}$$

$$(i)\,\{\mathscr{S}^e\lambda = -\mathbf{n_1}\cdot\nabla\mathbf{z_1} - \mathbf{n_2}\cdot\nabla\mathbf{w}, \quad (ii)\begin{cases} -\Delta w \geq f^i \text{ in } \Omega^i, \\ w \geq \psi \text{ in } \Omega^i, \\ w = 0 \text{ on } \partial\Omega \setminus \Gamma, \\ w = \lambda \text{ on } \Gamma, \end{cases}$$

$$\begin{cases} -\Delta z_2 = 0 \text{ in } \Omega^e, \\ z_2 = 0 \text{ on } \partial\Omega \setminus \Gamma, \\ z_2 = \lambda \text{ on } \Gamma, \end{cases}$$

The resulting solution in Ω^e, is $u|_{\Omega^e} = z = z_1 + z_2$. The solutions of $(i),(ii)$, i.e. λ and w can be approximated in an iterative manner by using a fixed point iteration (see section 2.3). The weak formulations of the above problems are given below.

$$\begin{cases} \text{find } z_1 \in H_0^1(\Omega^e) \text{ such that } \forall v \in H_0^1(\Omega^e), \\ a^e(z_1, v) = (f^e, v)_{\Omega^e}, \end{cases} \tag{9a}$$

$$\begin{cases} \text{find } \lambda \in \Lambda \text{and } w \in E\lambda + K^i \text{ such that } \forall \mu \in \Lambda \text{and } v \in K^i, \\ (\mathscr{S}^e\lambda, \mu) = ((f^e, F^e\mu^e) - a^e(z_1, F^e\mu^e)) + ((f^i, F^i\mu^i) - a^i(w, F^i\mu^i)), \\ a^i(w, v - w) \geq (f^i, v - w)_{\Omega^i}, \end{cases} \tag{9b}$$

$$\begin{cases} \text{find } z_2 \in E\lambda + H_0^1(\Omega^e) \text{ such that } \forall v \in H_0^1(\Omega^e), \\ a^e(z_2, v) = 0, \end{cases} \tag{9c}$$

where $K^i = \{v \in V := H_0^1(\Omega^i) :\ v \ \geq \ \psi\}$.

2.1 Finite element discretization

Let $\Omega \subset \mathbb{R}^2$ be a bounded open convex subset and let $\mathfrak{T}_h$ be a conforming isotropic subdivision of $\overline{\Omega}$ into simplices $\mathbf{t}$. Let V_h^e, V_h^i denote the spaces of continuous piece-wise polynomials defined on the corresponding subdivision of Ω^e, Ω^i.

$$K_h^e := \{v_h \in V_h^e : v_h|_{\partial\Omega^e \cap \partial\Omega} = 0\}, \quad K_h^i := \{v_h \in V_h^i : v_h \geq \psi,\ v|_{\partial\Omega^i \cap \partial\Omega} = 0\}.$$

Let $\mathscr{N}^e, \mathscr{N}^i, \mathscr{N}^\Gamma$ denote the sets of nodes located, respectively, in the subdomains Ω^e, Ω^i and on the interface Γ. Let

$$K_h^e = \text{span}\{\phi_k,\ k \in \mathscr{N}^e\}, \ \ K_h^i = \text{span}\{\phi_k,\ k \in \mathscr{N}^i\} \ \ , K_h^\Gamma = \text{span}\{\phi_k,\ k \in \mathscr{N}^\Gamma\}$$

and let

$$S^h = \text{span}\{\gamma_0(\Gamma)\phi_k,\ k \in \mathscr{N}_i^\Gamma\}.$$

By using above definitions, we have the following finite element discretization for the two-domains method:

$$\begin{cases} \text{find } z_1^h \in K_h^e \ \ \forall v_h \in K_h^e \\ a^e(z_1^h, v_h) = (f^e, v_h), \end{cases} \tag{10}$$

$$\begin{cases} \text{find } \lambda_h \in S^h \text{ and } w_h \in K_h^i \text{ such that } \forall v_h \in K_h^i, \ \forall\ \mu_h \in S^h, \\ (\mathscr{S}^e \lambda_h, \mu_h) = ((f^e, F^e \mu_h^e) - a^e(z_1^h, F^e \mu_h^e)) + ((f^i, F^i \mu_h^i) - a^i(w_h, F^i \mu_h^i)), \\ a^i(w_h, v_h - w_h) \geq (f^i, v_h - w_h) \end{cases} \tag{11}$$

$$\begin{cases} \text{find } z_2^h = (E\lambda)_h + K_h^e \text{ such that } \ \forall v_h \in K_h^e, \\ a^e(z_h, v_h) = 0. \end{cases} \tag{12}$$

2.2 Matrix formulation

To obtain the matrix formulation of the above discrete formulation of the domain decomposition problem let us denote the unknown vectors by $\mathbf{u}^e, \mathbf{u}^i, \mathbf{u}^\Gamma$ and the right hand side vectors by $\mathbf{f}^e, \mathbf{f}^i, \mathbf{f}_\Gamma$ of lengths N^e, N^i, N^Γ respectively, such that $N = N^e + N^i + N^\Gamma$, with $A \in \mathbb{R}^{N\times N}$ and $\mathbf{f} \in \mathbb{R}^N$. Then the matrix representation of (1) can be written as

$$\begin{cases} A\mathbf{u} \ \geq \mathbf{f}, \\ \ \mathbf{u} \ \ \geq \Psi, \end{cases}$$

subject to the complementarity conditions $(\mathbf{f} - A\mathbf{u})_j(\mathbf{u} - \Psi)_j = 0$, with

$$\begin{pmatrix} A_{II}^e & O & A_{I\Gamma}^e \\ O & A_{II}^i & A_{I\Gamma}^i \\ A_{\Gamma I}^e & A_{\Gamma I}^i & A_{\Gamma\Gamma} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} \mathbf{u}_I^e \\ \mathbf{u}_I^i \\ \mathbf{u}_\Gamma \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}_I^e \\ \mathbf{f}_I^i \\ \mathbf{f}_\Gamma \end{pmatrix}, \tag{13}$$

where we have partitioned the degrees of freedom into those internal to Ω^e and to Ω^i and those on the interface Γ. By using this notation, the above discrete weak formulations have the following matrix form

$$A_{II}^e \mathbf{u}_I^{e,1} = \mathbf{f}_I^e, \tag{14a}$$

$$S^e \mathbf{u}_\Gamma = \mathbf{f}_\Gamma - A_{\Gamma I}^e \mathbf{u}_I^{e,1} - A_{\Gamma I}^i \mathbf{u}_I^i, \tag{14b}$$

$$A_{II}^i \mathbf{u}_I^i \ \geq \ \mathbf{f}_I^i - A_{I\Gamma}^i \mathbf{u}_\Gamma, \tag{14c}$$

$$A_{II}^e \mathbf{u}_I^{e,2} = -A_{I\Gamma}^e \mathbf{u}_\Gamma, \tag{14d}$$

subject to conditions $(\mathbf{f}_I^i - A_{II}^i \mathbf{u}_I^i - A_{I\Gamma}^i \mathbf{u}_\Gamma)_j (\mathbf{u}_I^i - \Psi_I)_j = 0$, which represent the complementarity conditions for (14c).

The set of equations (14a)-(14d) could be seen as a partial Schur complement approach for the system (13). The solutions $\mathbf{u}_I^i$ and $\mathbf{u}_\Gamma$ will be approximated in an iterative manner. The resulting solution is then $[\mathbf{u}_I^{e,1} + \mathbf{u}_I^{e,2}, \mathbf{u}_I^i, \mathbf{u}_\Gamma]$.

2.3 Domain decomposition algorithm

Equations (14b) and (14c) form a coupled system which we solve by using a fixed point iteration. We note here that, given $\mathbf{u}_I^i$, the solution of (14b) involving the Schur complement matrix S^e can be implemented by using a Krylov subspace solver with domain decomposition preconditioning, corresponding to some partition of Ω^e into several subdomains. On the other hand, (14c) is a standard linear complementarity problem posed on a small subdomain Ω^i. The proposed algorithm is included below.

Picard reduced QP algorithm

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step 0: Find an initial guess by using coarse mesh solution
step 1: find $\mathbf{u}_I^{e\{1\}} = (A_{II}^e)^{-1}\mathbf{f}_I^e$,
step 2:
for $k = 0, 1, 2, \ldots$, till convergence **do**
  Solve $S^e(\mathbf{u}_\Gamma)^{k+1} = (\mathbf{f}_\Gamma - A_{\Gamma I}^e \mathbf{u}_I^{e\{1\}} - A_{\Gamma I}^i (\mathbf{u}_I^i)^k)$
  Find $(\mathbf{u}_I^i)^{k+1} \in K^i$ such that

$$J((\mathbf{u}_I^i)^{k+1}) \leq J(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{K^i}$$

  where

$$J(\mathbf{v}) := \frac{1}{2}(\mathbf{v})^T A_{II}^i \mathbf{v} - (\mathbf{v})^T (\mathbf{f}_I^i - A_{I\Gamma}^i \mathbf{u}_\Gamma^{k+1})$$

  If converged, set $\mathbf{u}_\Gamma = \mathbf{u}_\Gamma^{k+1}$ and exit
end for
step 3: Compute

$$\mathbf{u}_I^{e\{2\}} = -(A_{II}^e)^{-1} A_{I\Gamma}^e \mathbf{u}_\Gamma$$

The resulting solution is then

$$\mathbf{u} = [\mathbf{u}_I^{e\{1\}} + \mathbf{u}_I^{e\{2\}}, \mathbf{u}_I^i, \mathbf{u}_\Gamma].$$

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3 Numerical Experiments

Test 1:One obstacle

For our first test problem, we consider an elastic membrane which lies above an obstacle of height 1 centered at the origin with square cross-section with side length $\ell^o = 0.3$ under the forcing function $f = 1$ with $\Omega = (-1,1)^2$. We choose Ω^i to be a square region with side-length ℓ^i which contains the support of the obstacle such that the interface boundary Γ lies outside of the obstacle support. In the given algorithm we solved PDI, in the step 2(ii) by using the matlab function `quadprog`, a built-in quadratic programming solver. The PDI is coupled together with the interface equality problem in step 2(i) in an iterative manner. The relation to constrained minimization problems with quadratic programming problem can be found in [8].

We apply fixed point DD algorithm with global complementarity condition as a stopping criterion $\max_{1\le i\le n} \mid (\text{Ł}\mathbf{u}-\mathbf{f})_i(\mathbf{u}-\Psi)_i \mid \le 10^{-3}$. The initial guess was computed on a fixed coarse mesh with n_0 nodes. Note that the variational inequality problem is now posed over a small subdomain, and hence has low complexity - we therefore decided not to report on it. Table 1 displays the number of fixed point iterations required to solve the coupled equations (14b), (14c). We see that the number of iterations grows logarithmically as we increase the level of refinement. On the other hand, reducing the size of Ω^i leads to a smaller number of iterations, while preserving the dependence behaviour on the refinement level.

Table 1 Fixed point iterations for test problem 1.

ℓ^i=	0.4	0.5	0.6
n = 1,089	8	10	10
4,225	12	16	17
16,641	17	25	26

Test 2: Three obstacles

For the same domain Ω we consider the obstacle problem with three square obstacles of height 1 with centers located at $(0.5,0.5),(-0.5,0.5),(0,-0.5)$ and equal sides $\ell^o = 0.3$. We performed the same investigation, where we chose Ω^i to be a multiply-connected domain consisting of square regions of side-length ℓ^i (see Fig. 1. The numerical results are displayed in Table 2. For this harder problem, the number of iterations displays a logarithmic dependence for ℓ^i sufficiently small, but deteriorates for larger Ω^i. However, this is not the context we devised our algorithm for. Finally, we note that for this test problem the variational inequality in step (ii) decouples into three independent variational inequalities.

Table 2 Fixed point iterations for test problem 2.

ℓ^i=	0.4	0.5	0.6
n = 1,089	9	14	14
4,225	14	21	24
16,641	19	32	38

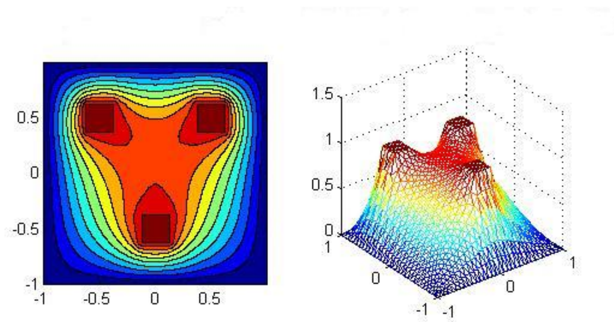

Fig. 1 Test problem 2: the choice of Ω^i for $\ell^i = 0.4$ and the corresponding solution.

4 Summary and future work

We described an algorithm for the solution of obstacle problems using a two-domain formulation. In the larger subdomain we solved a PDE, while in the smaller region containing the coincidence set we solved a variational inequality using a minimization formulation. The solution of the PDE, as well as the solution involving a reduced Schur complement problem can in practice be achieved via a parallel implementation of a Krylov method coupled with a domain decomposition preconditioner. Work in progress includes a Newton-Krylov solution of the non-linear problem (8). Future work is expected to include results validating this approach as well as an analysis of our algorithm. We are also interested to implement this method on general elliptic and parabolic problems.

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