A Domain-Based Multinumeric Method for the Steady-State Convection-Diffusion Equation

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1 Introduction

In the simulation of flow and transport of hydrocarbons in reservoirs, locally mass conservative methods are preferred. Methods that do not satisfy this property, will produce numerical mass errors that accumulate and will yield an unstable solution. Currently, finite volume methods are popular numerical methods in the oil industry. While they are computationally efficient, they are only of first order. Convergence of cell-centered finite volume solutions is theoretically obtained on specially constructed grids (such as Voronoi meshes) and for problems with no mixed second derivatives [3, 4, 8, 12, 6]. Discontinuous Galerkin methods also belong to the class of locally mass conservative methods. In addition, their flexibility allows for the use of complicated geometries, unstructured meshes, varying polynomial degrees and discontinuous coefficients. Discontinuous Galerkin solutions are accurate but their cost can be large as it is proportional to the the number of mesh elements (also called cells). In this paper, discontinuous Galerkin methods are used in certain parts of the domain whereas the cell-centered finite volume method is used in other parts. The model problem is a convection-diffusion problem in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$.

$$-\nabla \cdot (K\nabla u - \beta u) = f, \quad \text{in} \quad \Omega, \tag{1}$$
$$u = g, \quad \text{on} \quad \partial\Omega. \tag{2}$$

The spatially dependent coefficient K is bounded below and above by positive constants k_0 and k_1 respectively. The convective vector β is assumed to be divergence-free: $\nabla \cdot \beta = 0$.

The computational domain is partitioned into several subdomains. On each subdomain, either a discontinuous Galerkin method is used or a cell-centered finite volume is used. The advantage of a multinumeric approach lies in the ability of choosing a particular scheme for a particular subdomain. The discontinuous Galerkin method can yield accurate solutions in parts of the domain where the permeability of the porous medium varies over several orders of magnitude or in parts of the domain where anisotropy is important. In this work, the coupling of the two discretizations is done weakly by interface conditions. Two equivalent formulations are presented: a monolithic approach and an hybridized approach with Lagrange multipliers. This

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This work was partially funded by NSF and NHARP.

paper extends the result of [2] where the elliptic problem is analyzed. In [11], we apply the method to a transport equation. The idea of using different discretizations in different subdomains is well studied in the literature. For instance, the reader can refer to [1, 5, 10, 7].

An outline of the paper is the following. Section 2 defines first the discontinuous Galerkin and finite volume discretizations in each subdomain, then the coupling of the subdomains. Section 3 states the convergence of the method. Conclusions follow.

2 A multinumeric approach

The domain Ω is subdivided into non-overlapping subdomains Ω^i_{FV} and Ω^i_{DG}. Our proposed multinumerics scheme uses a finite volume method (FV) on the union of Ω^i_{FV}, denoted by Ω_F, and a discontinuous Galerkin (DG) method on the union of Ω^i_{DG}, denoted by Ω_D. Let $\mathscr{E}^h_D$ (resp. $\mathscr{E}^h_F$) be a subdivision of Ω_D (resp. Ω_F) made of cells V (Voronoi cells in Ω_F and either triangles/tetrahedra/hexaedra or Voronoi cells in Ω_D). We also denote by h_F (resp. h_D) the maximum diameter over all cells in Ω_F (resp. Ω_D) and we let $h = \max(h_F, h_D)$. We assume that the meshes match at the interface Γ_{DF} defined as:

$$\Gamma_{\mathrm{DF}} = \cup_i(\partial\Omega^i_{DG} \cap \partial\Omega^i_{FV})$$

The definition of the mesh $\mathscr{E}^h_F$ requires further notation. It is assumed that $\mathscr{E}^h_F$ is an admissible finite volume mesh, in the following sense:
(i) There is a family of nodes $\{x_V : V \in \mathscr{E}^h_F\}$ such that x_V belongs to $\overline{V}$ and if a face γ is shared by two neighboring cells V and W, it is assumed that x_W and x_V are distinct, and that the straight line going through x_V and x_W is orthogonal to γ.
(ii) For any boundary face $\gamma = \partial V \cap \partial\Omega$ for some V in $\mathscr{E}^h_F$, it is assumed that x_V does not lie on γ. However this condition can be relaxed.
We denote by $\Gamma^{h,\mathscr{I}}_{\mathrm{F}}$ the set of faces that belong to the interior of Ω_F and by $\Gamma^{h,\partial}_{\mathrm{F}}$ the set of boundary faces that belong to $\cup_i(\partial\Omega^i_{FV} \cap \partial\Omega)$. Similarly, the sets of interior and boundary faces of Ω_D are denoted by $\Gamma^{h,\mathscr{I}}_{\mathrm{D}}$ and $\Gamma^{h,\partial}_{\mathrm{D}}$ respectively. We also define $\Gamma^h_{\mathrm{F}} = \Gamma^{h,\mathscr{I}}_{\mathrm{F}} \cup \Gamma^{h,\partial}_{\mathrm{F}}$ and $\Gamma^h_{\mathrm{D}} = \Gamma^{h,\mathscr{I}}_{\mathrm{D}} \cup \Gamma^{h,\partial}_{\mathrm{D}}$. There remains the set of faces that belong to the interface Γ_{DF}; this particular set will be denoted by Γ^h_{DF}. We further decompose the boundary of Ω into inflow and outflow boundaries. The unit normal vector outward of Ω is denoted by n.

$$\Gamma^{h,\partial-}_{\mathrm{D}} = \{x \in \Gamma^{h,\partial}_{\mathrm{D}}, \quad \beta\cdot n \le 0\}, \quad \Gamma^{h,\partial+}_{\mathrm{D}} = \Gamma^{h,\partial}_{\mathrm{D}} \setminus \Gamma^{h,\partial-}_{\mathrm{D}}.$$

$$\Gamma^{h,\partial-}_{\mathrm{F}} = \{x \in \Gamma^{h,\partial}_{\mathrm{F}}, \quad \beta\cdot n \le 0\}, \quad \Gamma^{h,\partial+}_{\mathrm{F}} = \Gamma^{h,\partial}_{\mathrm{F}} \setminus \Gamma^{h,\partial-}_{\mathrm{F}}.$$

We now define a parameter d_γ that is associated to each face γ in $\Gamma^h_{\mathrm{F}} \cup \Gamma^h_{\mathrm{DF}}$. If the face γ is an interior face shared by two cells V and W in $\mathscr{E}^h_F$, the parameter d_γ is the

Euclidean distance between the nodes x_V and x_W: $d_\gamma = d(x_V, x_W)$. If the face γ is a boundary face ($\gamma \subset \partial V \cap \partial \Omega$), the parameter d_γ is the distance between the node x_V and the face γ, in other words $d_\gamma = d(x_V, y_\gamma)$, where y_γ denotes the non-empty intersection between the straight line going through x_V and orthogonal to γ. Finally, if the face γ lies on the interface Γ_{DF} and is shared by a cell V in $\mathscr{E}_F^h$ and a cell W in $\mathscr{E}_D^h$, the parameter d_γ is defined to be the distance between the node x_V and the edge γ. As in the boundary case, we can denote by y_γ the intersection between the straight line going through x_V and perpendicular to γ. Then, we have $d_\gamma = d(x_V, y_\gamma)$.

An admissible mesh in the finite volume regions is such that there is some positive number $\theta > 0$ such that

$$
\begin{aligned}
d_\gamma \geq \theta \max(h_V, h_W), \quad \forall \gamma \in \Gamma_{\mathrm{F}}^{h,\mathscr{I}}, \quad \gamma = \partial V \cap \partial W,\\
d_\gamma \geq \theta h_V, \quad \forall \gamma \in \Gamma_{\mathrm{F}}^{h,\partial}, \quad \gamma = \partial V \cap \partial \Omega,\\
d_\gamma \geq \theta h_V, \quad \forall \gamma \in \Gamma_{\mathrm{DF}}^{h}, \quad \gamma = \partial V \cap \partial W, \quad V \in \mathscr{E}_F^h, W \in \mathscr{E}_D^h.
\end{aligned}
$$

A standard harmonic average of the diffusion coefficient K is now defined:

$$
\begin{aligned}
K_\gamma = d_\gamma \left| \int_{x_V}^{x_W} \frac{ds}{K(s)} \right|^{-1}, \quad \forall \gamma \in \Gamma_{\mathrm{F}}^{h,\mathscr{I}}, \quad \gamma = \partial V \cap \partial W,\\
K_\gamma = d_\gamma \left| \int_{x_V}^{y_\gamma} \frac{ds}{K(s)} \right|^{-1}, \quad \forall \gamma \in \Gamma_{\mathrm{F}}^{h,\partial}, \quad \gamma = \partial V \cap \partial \Omega,\\
K_\gamma = d_\gamma \left| \int_{x_V}^{y_\gamma} \frac{ds}{K(s)} \right|^{-1}, \quad \forall \gamma \in \Gamma_{\mathrm{DF}}^{h}, \quad \gamma = \partial V \cap \partial W, \quad V \in \mathscr{E}_F^h, W \in \mathscr{E}_D^h.
\end{aligned}
$$

It is easy to see that K_γ is bounded above and below by k_1 and k_0 respectively. We denote by $|\gamma|$ the measure of the face γ.

Let X^{DG} be the space of discontinuous piecewise polynomials of degree $r \geq 1$ in the DG subdomains. Let X^{FV} be the space of piecewise constants in the FV subdomains. The restriction of the numerical solution to the DG subdomains (resp. FV subdomains) is denoted by u_{DG} (resp. u_{FV}).

2.1 Bilinear Forms

The differential operators are discretized by an interior penalty discontinuous Galerkin method in some subdomains and by a cell-centered finite volume method in other subdomains.

First, we define the jump of any discontinuous piecewise polynomial function. For any face γ, we fix a unit normal vector n_γ to γ. If γ is a boundary face, then n_γ is the outward normal to Ω. If γ belongs to the interface Γ_{DF}^h, then the vector n_γ is chosen to point from the DG region into the FV region. In the definition of the jump $[v]$ of a function v given below, we assume that the face γ is shared by two cells V and W, and that the normal vector n_γ points from V into W. For the interior faces, we define

$$[v]|_\gamma = v|_V - v|_W, \quad \gamma \in \Gamma_{\mathrm{F}}^{h,\mathscr{I}} \cup \Gamma_{\mathrm{D}}^{h,\mathscr{I}}, \quad \gamma = \partial V \cap \partial W$$

For the boundary faces, we define

$$[v]|_\gamma = v|_V, \quad \gamma \in \Gamma_{\mathrm{F}}^{h,\partial} \cup \Gamma_{\mathrm{D}}^{h,\partial}, \quad \gamma = \partial V \cap \partial \Omega.$$

In the definitions above, it is understood that $v|_W = v(x_W)$ if W is a cell in the FV subdomains.

The average of a discontinuous function v on a face is denoted by $\{v\}$ and defined below:

$$\{v\}|_\gamma = \frac{1}{2}(v|_V + v|_W), \quad \forall \gamma = \partial V \cap \partial W,$$
$$\{v\}|_\gamma = v|_V, \quad \forall \gamma = \partial V \cap \partial \Omega.$$

Finally we define the upwind $v^\uparrow$ on the faces. For a given face γ in $\Gamma_{\mathrm{D}}^h \cup \Gamma_{\mathrm{F}}^h \cup \Gamma_{\mathrm{DF}}^h$ shared by cells V and W such that n_γ points from V into W, we have

$$v^\uparrow = \begin{cases} v|_V & \text{if} \quad \beta \cdot n_\gamma \geq 0, \\ v|_W & \text{if} \quad \beta \cdot n_\gamma < 0. \end{cases}$$

In what follows, we derive the bilinear forms corresponding to each subdomain. First, we multiply (1) by a function $v \in X^{DG}$, integrate over one DG cell V:

$$\int_V (K\nabla u - \beta u) \cdot \nabla v - \int_{\partial V} (K\nabla u - \beta u) \cdot n_V v = \int_V f v$$

We sum over all the cells in all the DG subdomains, use the definition of the normal vector n_γ and the regularity of the exact solution to obtain:

$$\sum_{V \in \mathscr{E}_D^h} \int_V (K\nabla u - \beta u) \cdot \nabla v - \sum_{\gamma \in \Gamma_{\mathrm{D}}^{h,\mathscr{I}}} \int_\gamma (\{K\nabla u\} - \beta u^\uparrow) \cdot n_\gamma [v]$$
$$- \sum_{\gamma \in \Gamma_{\mathrm{D}}^{h,\partial} \cup \Gamma_{\mathrm{DF}}^h} \int_\gamma (K\nabla u - \beta u) \cdot n_\gamma v = \sum_{V \in \mathscr{E}_D^h} \int_V f v$$

Stabilization terms are added for the interior penalty discontinuous Galerkin method. The penalty parameter is denoted by $\sigma > 0$ and the symmetrization parameter by $\varepsilon \in \{-1, +1\}$. The penalty parameter is assumed to be large enough if $\varepsilon = -1$ and is taken equal to 1 if $\varepsilon = +1$. The parameter h_γ denotes the maximum diameter of the neighboring cells V and W, that share the face γ.

$$\sum_{V \in \mathscr{E}_D^h} \int_V (K\nabla u - \beta u) \cdot \nabla v - \sum_{\gamma \in \Gamma_{\mathrm{D}}^h} \int_\gamma \big(\{K\nabla u \cdot n_\gamma\}[v] - \varepsilon \{K\nabla v \cdot n_\gamma\}[u]\big)$$

$$+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h}}\sigma h_\gamma^{-1}\int_\gamma[u][v]+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\mathscr{I}}}\int_\gamma\beta\cdot n_\gamma u^{\uparrow}[v]+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\partial}}\int_\gamma\beta\cdot n_\gamma uv$$
$$-\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_\gamma(K\nabla u-\beta u)\cdot n_\gamma v=\sum_{V\in\mathscr{E}_D^h}\int_V fv+\varepsilon\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\partial}}\int_\gamma\{K\nabla v\cdot n_\gamma\}g$$
$$+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\partial}}\sigma h_\gamma^{-1}\int_\gamma gv$$

From this derivation, we define the bilinear form for the DG subdomains as:

$$a_{\mathrm{DG}}(u,v)=\sum_{V\in\mathscr{E}_D^h}\int_V(K\nabla u-\beta u)\cdot\nabla v-\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h}}\int_\gamma\big(\{K\nabla u\cdot n_\gamma\}[v]-\varepsilon\{K\nabla v\cdot n_\gamma\}[u]\big)$$
$$+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h}}\sigma h_\gamma^{-1}\int_\gamma[u][v]+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\mathscr{I}}}\int_\gamma\beta\cdot n_\gamma u^{\uparrow}[v]+\sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\partial+}}\int_\gamma\beta\cdot n_\gamma uv$$
$$+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma u(y_\gamma)v(y_\gamma)+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_{\gamma+}\beta\cdot n_\gamma uv$$

In the last term, the subset of a face γ on which $\beta\cdot n_\gamma$ is non-negative is denoted by $\gamma+$. This corresponds to the outflow part of the face. The inflow part is denoted by $\gamma-$.

Second, we multiply (1) by a function $v\in X^{FV}$, that is piecewise constant, integrate over one FV cell V:

$$-\int_{\partial V}(K\nabla u-\beta u)\cdot n_V v=\int_V fv$$

We sum over all the FV cells and use the regularity of the exact solution:

$$\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\int_\gamma(-K\nabla u\cdot n_\gamma+\beta\cdot n_\gamma u^{\uparrow})[v]+\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\partial}}\int_\gamma(-K\nabla u\cdot n_\gamma+\beta\cdot n_\gamma u)v$$
$$+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_\gamma(K\nabla u-\beta u)\cdot n_\gamma v=\sum_{V\in\mathscr{E}_F^h}\int_V fv$$

A cell-centered finite difference approximation is used to approximate the flux across the faces. Therefore we define the bilinear form in the FV regions as:

$$a_{\mathrm{FV}}(u,v)=\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma[u][v]+\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\int_\gamma\beta\cdot n_\gamma u^{\uparrow}[v]+\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\partial+}}\int_\gamma\beta\cdot n_\gamma uv$$
$$+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma uv-\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_{\gamma-}\beta\cdot n_\gamma uv$$

Finally the source function f and the boundary conditions are handled by the following bilinear forms:

$$\ell_{DG}(v) = \int_{\Omega_D} fv + \varepsilon \sum_{\gamma\in\Gamma_{\rm D}^{h,\partial}} \int_\gamma K\nabla v\cdot n_\gamma g + \sum_{\gamma\in\Gamma_{\rm D}^{h,\partial}} \sigma h_\gamma^{-1} \int_\gamma gv - \sum_{\gamma\in\Gamma_{\rm D}^{h,\partial-}} \int_\gamma \beta\cdot n_\gamma gv$$

$$\ell_{FV}(v) = \int_{\Omega_F} fv + \sum_{\gamma\in\Gamma_{\rm F}^{h,\partial}} \frac{|\gamma|}{d_\gamma} K_\gamma g(y_\gamma)v - \sum_{\gamma\in\Gamma_{\rm F}^{h,\partial-}} \int_\gamma \beta\cdot n_\gamma gv.$$

2.2 A monolithic formulation

The definition of the multinumeric scheme, without Lagrange multipliers, is given in this section. Existence and uniqueness of the solution is shown.

The numerical method is as follows: find $u_{DG}\in X^{DG}$, $u_{FV}\in X^{FV}$ such that

$$a_{DG}(u_{DG},v_{DG}) = \ell_{DG}(v_{DG}) + \sum_{\gamma\in\Gamma_{\rm DF}^h} \frac{|\gamma|}{d_\gamma} K_\gamma u_{FV} v_{DG}(y_\gamma) - \sum_{\gamma\in\Gamma_{\rm DF}^h} \int_{\gamma-} \beta\cdot n_\gamma u_{FV} v_{DG}, \quad (3)$$

$$a_{FV}(u_{FV},v_{FV}) = \ell_{FV}(v_{FV}) + \sum_{\gamma\in\Gamma_{\rm DF}^h} \frac{|\gamma|}{d_\gamma} K_\gamma u_{DG}(y_\gamma) v_{FV} + \sum_{\gamma\in\Gamma_{\rm DF}^h} \int_{\gamma+} \beta\cdot n_\gamma u_{DG} v_{FV}, \quad (4)$$

for all $v_{DG}\in X^{DG}$ and all $v_{FV}\in X^{FV}$.

Lemma 1. *There exists a unique solution (u_{DG},u_{FV}), satisfying (3)-(4).*

Proof. Let us assume that $f=g=0$ and take $v_{DG}=u_{DG}$ and $v_{FV}=u_{FV}$ in (3)-(4). We have

$$a_{DG}(u_{DG},u_{DG}) + a_{FV}(u_{FV},u_{FV}) = 2\sum_{\gamma\in\Gamma_{\rm DF}^h} \frac{|\gamma|}{d_\gamma} K_\gamma u_{FV} u_{DG}(y_\gamma) + \sum_{\gamma\in\Gamma_{\rm DF}^h} \int_\gamma |\beta\cdot n_\gamma| u_{DG} u_{FV}.$$

We expand the DG form:

$$\begin{aligned} a_{\rm DG}(u_{DG},u_{DG}) = &\sum_{V\in\mathscr{E}_D^h} \|K^{1/2}\nabla u_{DG}\|^2_{L^2(V)} + \sum_{\gamma\in\Gamma_{\rm D}^h} \sigma h_\gamma^{-1} \|[u_{DG}]\|^2_{L^2(\gamma)} \\ &+ \sum_{\gamma\in\Gamma_{\rm DF}^h} \frac{|\gamma|}{d_\gamma} K_\gamma u_{DG}(y_\gamma)^2 - (1-\varepsilon)\sum_{\gamma\in\Gamma_{\rm D}^h} \int_\gamma \{K\nabla u_{DG}\cdot n_\gamma\}[u_{DG}] - \sum_{V\in\mathscr{E}_D^h} \int_V \beta u_{DG}\cdot\nabla u_{DG} \\ &+ \sum_{\gamma\in\Gamma_{\rm D}^{h,\mathscr{I}}} \int_\gamma \beta\cdot n_\gamma u_{DG}^{\uparrow}[u_{DG}] + \sum_{\gamma\in\Gamma_{\rm D}^{h,\partial+}} \int_\gamma \beta\cdot n_\gamma u_{DG}^2 + \sum_{\gamma\in\Gamma_{\rm DF}^h} \int_{\gamma+} \beta\cdot n_\gamma u_{DG}^2 \end{aligned}$$

Using standard techniques to DG methods [9], one can show that

$$
\begin{aligned}
-\sum_{V\in\mathscr{E}_D^h}\int_V \beta u_{DG}\cdot\nabla u_{DG} + \sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\mathscr{I}}}\int_\gamma \beta\cdot n_\gamma u_{DG}^{\uparrow}[u_{DG}] + \sum_{\gamma\in\Gamma_{\mathrm{D}}^{h,\partial+}}\int_\gamma \beta\cdot n_\gamma u_{DG}^2 \\
= \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{D}}^h} \||\beta\cdot n_\gamma|^{1/2}[u_{DG}]\|_{L^2(\gamma)}^2 - \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{DF}}^h}\int_\gamma \beta\cdot n_\gamma u_{DG}^2
\end{aligned}
$$

In addition, we can show that there is a constant $M>0$ independent of h such that

$$
\begin{aligned}
\sum_{V\in\mathscr{E}_D^h}\|K^{1/2}\nabla u_{DG}\|_{L^2(V)}^2 + \sum_{\gamma\in\Gamma_{\mathrm{D}}^h}\sigma h_\gamma^{-1}\|[u_{DG}]\|_{L^2(\gamma)}^2 - (1-\varepsilon)\sum_{\gamma\in\Gamma_{\mathrm{D}}^h}\int_\gamma\{K\nabla u_{DG}\cdot n_\gamma\}[u_{DG}] \\
\geq M\left(\sum_{V\in\mathscr{E}_D^h}\|K^{1/2}\nabla u_{DG}\|_{L^2(V)}^2 + \sum_{\gamma\in\Gamma_{\mathrm{D}}^h}\sigma h_\gamma^{-1}\|[u_{DG}]\|_{L^2(\gamma)}^2\right)
\end{aligned}
$$

For the FV bilinear form, we have

$$
\begin{aligned}
a_{\mathrm{FV}}(u_{FV},u_{FV}) = \sum_{\gamma\in\Gamma_{\mathrm{F}}^h}\frac{|\gamma|}{d_\gamma}K_\gamma[u_{FV}]^2 + \sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\int_\gamma \beta\cdot n_\gamma u_{FV}^{\uparrow}[u_{FV}] + \sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\partial+}}\int_\gamma \beta\cdot n_\gamma u_{FV}^2 \\
+ \sum_{\gamma\in\Gamma_{\mathrm{DF}}^h}\frac{|\gamma|}{d_\gamma}K_\gamma u_{FV}^2 - \sum_{\gamma\in\Gamma_{\mathrm{DF}}^h}\int_{\gamma-}\beta\cdot n_\gamma u_{FV}^2
\end{aligned}
$$

We observe that, if $u_{FV}^{\downarrow}$ denotes the downwind value of u_{FV}, we have

$$
\begin{aligned}
\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\int_\gamma \beta\cdot n_\gamma u_{FV}^{\uparrow}[u_{FV}] = \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\||\beta\cdot n_\gamma|^{1/2}[u_{FV}]\|_{L^2(\gamma)}^2 \\
+\frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\int_\gamma|\beta\cdot n_\gamma|((u_{FV}^{\uparrow})^2-(u_{FV}^{\downarrow})^2)
\end{aligned}
$$

Since β is divergence-free, we obtain

$$
\begin{aligned}
\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\int_\gamma \beta\cdot n_\gamma u_{FV}^{\uparrow}[u_{FV}] = \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\||\beta\cdot n_\gamma|^{1/2}[u_{FV}]\|_{L^2(\gamma)}^2 \\
-\frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\partial}}\int_\gamma \beta\cdot n_\gamma u_{FV}^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{DF}}^h}\int_\gamma \beta\cdot n_\gamma u_{FV}^2
\end{aligned}
$$

Combining the results above yields

$$
M\sum_{V\in\mathscr{E}_D^h}\|K^{1/2}\nabla u_{DG}\|_{L^2(V)}^2 + M\sum_{\gamma\in\Gamma_{\mathrm{D}}^h}\sigma h_\gamma^{-1}\|[u_{DG}]\|_{L^2(\gamma)}^2 + \frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{D}}^h}\||\beta\cdot n_\gamma|^{1/2}[u_{DG}]\|_{L^2(\gamma)}
$$

$$+\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma[u_{FV}]^2+\frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\mathscr{I}}}\||\beta\cdot n_\gamma|^{1/2}[u_{FV}]\|^2_{L^2(\gamma)}+\frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{F}}^{h,\partial}}\int_\gamma|\beta\cdot n_\gamma|u_{FV}^2$$

$$+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma(u_{DG}(y_\gamma)-u_{FV})^2+\frac{1}{2}\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_\gamma|\beta\cdot n_\gamma|(u_{DG}-u_{FV})^2\le 0$$

The inequality above immediately implies that u_{DG} and u_{FV} are zero everywhere. Thus, we have proved uniqueness of the solution. Since the finite-dimensional problem is linear, this is equivalent to showing existence of the solution.

2.3 Formulation with Lagrange multipliers

In this section, we rewrite the method (3)-(4) in a hybridized form for the elliptic problem. Lagrange multipliers are defined on the interface between the subdomains.

Let $\Lambda_h^0\subset L^2(\Gamma_{12})$ be the finite dimensional space of piecewise constants on the partition of Γ_{12}. Assume that the convection vector β is zero. The hybridized DG-FV scheme becomes: solve for $u_{\mathrm{DG}}\in X^{\mathrm{DG}}$, $u_{\mathrm{FV}}\in X^{\mathrm{FV}}$, $\lambda_{DG}\in\Lambda_h^0$, $\lambda_{FV}\in\Lambda_h^0$ satisfying

$$a_{DG}(u_{DG},v_{DG})=\ell_{DG}(v_{DG})+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma\lambda_{FV}v_{DG}(y_\gamma),\quad\forall v\in X^{DG}\tag{5}$$

$$a_{FV}(u_{FV},v_{FV})=\ell_{FV}(v_{FV})+\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\frac{|\gamma|}{d_\gamma}K_\gamma\lambda_{DG}v_{FV},\quad\forall v\in X^{FV}\tag{6}$$

$$\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_\gamma(\lambda_{DG}-u_{DG}(y_\gamma))\mu=0,\quad\forall\mu\in\Lambda_h^0\tag{7}$$

$$\sum_{\gamma\in\Gamma_{\mathrm{DF}}^{h}}\int_\gamma(\lambda_{FV}-u_{FV})\mu=0,\quad\forall\mu\in\Lambda_h^0\tag{8}$$

Lemma 2. *There exists a unique solution to (5)-(8)*

Proof. To show uniqueness of the solution, we assume that $f=g=0$ and take $v_{DG}=u_{DG}$ and $v_{FV}=u_{FV}$ in (5) and (6). We observe that (7) and (8) imply that

$$\lambda_{DG}|_\gamma=u_{DG}(y_\gamma),\quad\lambda_{FV}|_\gamma=u_{FV},\quad\forall\gamma\in\Gamma_{\mathrm{DF}}^{h}$$

The rest of the proof follows the proof of Lemma 1.

3 Error analysis

In this section, convergence of the multinumeric approach is shown under some regularity assumptions of the exact solution.

Assume that the relative gradient of the exact solution near the interfaces with respect to the gradient in the DG subdomains is small. In particular, given a face $\gamma \in \Gamma^h_{\mathrm{DF}}$ that belongs to a DG cell denoted by V_γ, assume that there is a constant C independent of h_D such that

$$\Big(\sum_{\gamma \in \Gamma^h_{\mathrm{DF}}} \|\nabla u\|^2_{L^2(V_\gamma)}\Big)^{1/2} \le C h_D \Big(\sum_{V \in \mathscr{E}^h_D} \|\nabla u\|^2_{L^2(V)}\Big)^{1/2} \tag{9}$$

This assumption is an indicator on how to choose the interface. We want to place the interface where the exact solution does not vary as much as it does in the interior of the discontinuous Galerkin domain. In the simple case where the exact solution is linear and its gradient is uniformly constant, this assumption is not satisfied (see remark 1).

We recall that by convention, the jump $[u - u_{FV}]$ on an edge that belongs to Γ^h_{F} is the difference between $u(x_V) - u_{FV}(x_V)$ and $u(x_W) - u_{FV}(x_W)$ if the edge is shared by the Voronoi cells V and W.

Theorem 1. *Assume that u belongs to $H^2(\Omega)$ and that $u|_{\Omega_D}$ belongs to $H^{r+1}(\mathscr{E}^h_D)$, for $r \ge 1$. Under the assumption (9), there exists a constant C independent of h such that*

$$\sum_{V \in \mathscr{E}^h_D} \|K^{1/2}\nabla(u - u_{DG})\|^2_{L^2(V)} + \sum_{\gamma \in \Gamma^h_{\mathrm{D}}} \sigma h_\gamma^{-1} \|[u - u_{DG}]\|^2_{L^2(\gamma)} + \sum_{\gamma \in \Gamma^h_{\mathrm{F}}} \frac{\gamma}{d_\gamma} K_\gamma [u - u_{FV}]^2$$
$$+ \sum_{\gamma \in \Gamma^h_{\mathrm{D}}} \||\beta \cdot n_\gamma|^{1/2}[u - u_{DG}]\|^2_{L^2(\gamma)} + \sum_{\gamma \in \Gamma^h_{\mathrm{F}}} \||\beta \cdot n_\gamma|^{1/2}[u - u_{DG}]\|^2_{L^2(\gamma)}$$
$$+ \sum_{\gamma \in \Gamma^h_{\mathrm{DF}}} \frac{|\gamma|}{d_\gamma} K_\gamma (u_{DG}(y_\gamma) - u_{FV})^2 \le C(h_D^2 + h_F^2)$$

Proof. An outline of the proof is given. First we observe that the scheme (3)-(4) is not consistent because of the use of finite difference approximations in the FV subdomains and on the interfaces between the subdomains. We introduce an optimal approximation, $\tilde{u}$, of the exact solution such that $\tilde{u}|_{\Omega_D}$ (resp. $\tilde{u}|_{\Omega_F}$) belongs to X^{DG} (resp. X^{FV}). We define

$$\chi_{DG} = u_{DG} - \tilde{u}|_{\Omega_D}, \quad \chi_{FV} = u_{FV} - \tilde{u}|_{\Omega_F}, \quad \xi = u - \tilde{u}$$

An error equation can be obtained:

$$a_{DG}(\chi_{DG}, v_{DG}) + a_{FV}(\chi_{FV}, v_{FV}) + \sum_{\gamma \in \Gamma^h_{\mathrm{DF}}} \int_{\gamma-} \beta \cdot n_\gamma \chi_{FV} v_{DG} - \sum_{\gamma \in \Gamma^h_{\mathrm{DF}}} \frac{|\gamma|}{d_\gamma} K_\gamma \chi_{FV} v_{DG}(y_\gamma)$$

$$- \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \int_{\gamma+} \beta \cdot n_\gamma \chi_{DG} v_{FV} - \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \frac{|\gamma|}{d_\gamma} K_\gamma \chi_{DG}(y_\gamma) v_{FV} = a_{DG}(\xi_{DG}, v_{DG}) + a_{FV}(\xi_{FV}, v_{FV})$$
$$+ \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \int_{\gamma-} \beta \cdot n_\gamma \xi|_{\Omega_F} v_{DG} - \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \frac{|\gamma|}{d_\gamma} K_\gamma \xi|_{\Omega_F} v_{DG}(y_\gamma) - \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \int_{\gamma+} \beta \cdot n_\gamma \xi|_{\Omega_D} v_{FV}$$
$$- \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \frac{|\gamma|}{d_\gamma} K_\gamma \xi|_{\Omega_D}(y_\gamma) v_{FV} + R,$$

where R is a residual term resulting from the consistency error. An expression for R is:

$$R = \sum_{\gamma \in \Gamma_{\mathrm{F}}^h} R_\gamma(u)[v_{FV}] + \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \int_\gamma K \nabla u \cdot n_\gamma (v_{DG} - v_{DG}(y_\gamma))$$
$$+ \sum_{\gamma \in \Gamma_{\mathrm{DF}}^h} \int_\gamma R_\gamma(u)(v_{DG}(y_\gamma) - v_{FV}) \tag{10}$$

The residual quantities $R_\gamma(u)$ are defined on the interior faces of the FV subdomains as

$$R_\gamma(u) = -\int_\gamma K \nabla u \cdot n_\gamma - \frac{|\gamma|}{d_\gamma} K_\gamma (u(x_V) - u(x_W)), \quad \forall \gamma = \partial V \cap \partial W \quad \forall \gamma \in \Gamma_{\mathrm{F}}^{h,\mathscr{I}}$$

This expression is slightly modified for the exterior boundary faces of the FV subdomains:

$$R_\gamma(u) = -\int_\gamma K \nabla u \cdot n_\gamma - \frac{|\gamma|}{d_\gamma} K_\gamma (u(x_V) - g(y_\gamma)), \quad \forall \gamma = \partial V, \quad \forall \gamma \in \Gamma_{\mathrm{F}}^{h,\partial}$$

For the interfaces between the FV and DG subdomains, the residual term is defined as

$$R_\gamma(u) = -K \nabla u \cdot n_\gamma - \frac{K_\gamma}{d_\gamma} (u(y_\gamma) - u(x_W)), \quad \forall \gamma \in \partial W, W \in \mathscr{E}_F^h, \quad \forall \gamma \in \Gamma_{\mathrm{DF}}^h$$

Next, we choose $v_{DG} = \chi_{DG}$ and $v_{FV} = \chi_{FV}$ in the error equation. The error estimate follows by using trace inequalities, approximation results, and the following bounds on the residuals, that involve the Hessian matrix $\mathbf{H}(u)$ (see [3]):

$$|R_\gamma(u)|^2 \le C \frac{h_F^2 |\gamma|}{d_\gamma} \int_{\mathscr{V}_\gamma} |\mathbf{H}(u)|^2, \quad \forall \gamma \in \Gamma_{\mathrm{F}}^h$$
$$\left(\int_\gamma |R_\gamma(u)| \right)^2 \le C \frac{h_F^2 |\gamma|}{d_\gamma} \int_{\mathscr{V}_\gamma} |\mathbf{H}(u)|^2, \quad \forall \gamma \in \Gamma_{\mathrm{DF}}^h$$

The Hessian is integrated over the region $\mathscr{V}_\gamma$ defined by

$$\mathscr{V}_\gamma = \mathscr{V}_{W,\gamma} \cup \mathscr{V}_{V,\gamma}, \quad \forall \gamma = \partial V \cap \partial W$$

with

$$\mathscr{V}_{W,\gamma} = \{t x_W + (1-t)x : x \in \gamma, t \in [0,1]\}$$

Remark 1. If the assumption (9) is removed, the multinumeric approach converges suboptimally. Indeed, there is a loss of $h_D^{1/2}$ in the bound of the last term in the definition of the residual in (10).

4 Conclusions

Cell-centered finite volume methods use Voronoi cells for unstructured meshes. Discontinuous Galerkin methods converge on general mesh elements including Voronoi grids. In addition, for two-dimensional problems, Voronoi cells can naturally and easily be partitioned into triangles by using the underlying Delaunay triangulation. In this work, we formulate and analyze a method that couples DG and FV methods via mesh interfaces. One appealing feature of the method is that, once a Voronoi grid is built, the decomposition of the domain into DG regions and FV regions is done easily and this decomposition can vary with each simulation.

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