Hybrid dual-primal FETI-Schur complement method for Stokes

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1 Discrete Stokes

The algebraic Stokes is of the following saddle point form: Find $(U_h, P_h) \in V_h \times Q_h$ such that

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} U_h \\ P_h \end{pmatrix} = \begin{pmatrix} F_h \\ 0 \end{pmatrix}. \tag{1}$$

We suppose that the system (1) arises from the mixed finite-element discretization of Stokes on a domain Ω. We consider spaces V_h and Q_h that satisfy the inf-sup condition and whose elements are continuous. Such spaces can be found in [5] and include Hood-Taylor and Mini elements. Under the inf-sup condition and assuming a mixed boundary condition on the velocity there exists a unique solutuion to (1).

2 Hybrid dual-primal FETI-Schur

Stokes is a bottleneck in the analysis of incompressible fluid flows and is the subject of many researches. The numerical solution of the system (1) that arises from its discretization is a challenging problem because of the indefiniteness of saddle-point problems [1]. Memory space storage is an other important issue to deal with for large three-dimensional problems. An overview of solution methods to solve saddle-point problems is given in [1]. We focus on iterative methods such as FETI and BDD that save memory space and have proved efficiency for many linear systems. The domain Ω is split into N non-overlapping subdomains $\{\Omega^{(s)}\}_{s=1,\cdots,N}$ with interface $\Gamma_I = \cup_{s,q=1}^{N} \{\overline{\Omega}^{(s)} \cap \overline{\Omega}^{(q)}\}$. Degrees of freedom of each subdomain $\Omega^{(s)}$ are split into internal degrees of freedom designated by subscript i and degrees of freedom designated by subscript Γ that correspond to the interface of the subdomain $\Omega^{(s)}$ with other subdomains. Related to the spliting above, FETI and BDD split the original linear systems into subproblems whose solutions are flux and trace continuous respectively [4, 9]. FETI adresses these compatibility requirements by introducing a unique Lagrange multiplier on the interface to ensure the weak continuity of the subsolutions. FETI is dual to BDD that imposes a unique trace to the subsolutions on the interface. The original system is thus reduced in both cases to interface problems to be solved by Krylov methods that nullify the residual at convergence. The resid-

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uals in FETI and BDD are the jump of the solutions and of the flux on the interface respectively.These domain decomposition methods have been successfully extended to solve the system (1) when the discrete pressure is discontinuous. Their interface systems become mixed problems when the discrete velocity and pressure are both continuous. The spectral distribution of the interface operators slows down the rate of convergence of FETI and BDD that is proven to be optimal for systems arising from the discretization of elliptic problems. The interface unknowns resulting from the combination of FETI and BDD should be physically homogeneous [6] and well-suited for saddle-point problems such as (1) that arise in many applications [1]. We split the system (1) into N subsystems, renumber the unkwnows starting with the internal ones to get the following system:
Local systems

$$\begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix} \begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \end{pmatrix} + \begin{pmatrix} A_{i\Gamma} \\ B_{i\Gamma} \end{pmatrix} U_\Gamma^{(s)} + \begin{pmatrix} {B_{i\Gamma}^{(s)}}^T \\ 0 \end{pmatrix} P_\Gamma^{(s)} = \begin{pmatrix} F_i^{(s)} \\ 0 \end{pmatrix}, \tag{2}$$

interface problems

$$\begin{pmatrix} A_{\Gamma i}^{(s)} & {B_{i\Gamma}^{(s)}}^T \end{pmatrix} \begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \end{pmatrix} + A_{\Gamma\Gamma}^{(s)} U_\Gamma^{(s)} + {B_{\Gamma\Gamma}^{(s)}}^T P_\Gamma^{(s)} = F_\Gamma^{(s)}, \tag{3}$$

incompressibility conditions

$$\begin{pmatrix} B_{\Gamma i}^{(s)} & 0 \end{pmatrix} \begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \end{pmatrix} + B_{\Gamma\Gamma}^{(s)} U_\Gamma^{(s)} = 0, \quad s = 1, \cdots, N. \tag{4}$$

Systems (2)-(4) supplemented with continuity conditions on the velocity and on the pressure through the interface are equivalent to system (1).
Introduce notations:

$$
\begin{aligned}
M_{uu}^{(s)} &= A_{\Gamma\Gamma}^{(s)} - \begin{pmatrix} A_{\Gamma i}^{(s)} & {B_{i\Gamma}^{(s)}}^T \end{pmatrix} \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{i\Gamma}^{(s)} \\ B_{i\Gamma}^{(s)} \end{pmatrix}, \\
M_{up}^{(s)} &= {B_{\Gamma\Gamma}^{(s)}}^T - \begin{pmatrix} A_{\Gamma i}^{(s)} & {B_{i\Gamma}^{(s)}}^T \end{pmatrix} \begin{pmatrix} A_{ii}^{(s)} & {B_{\Gamma\Gamma}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} {B_{i\Gamma}^{(s)}}^T \\ 0 \end{pmatrix}, \\
M_{pu}^{(s)} &= B_{\Gamma\Gamma}^{(s)} - \begin{pmatrix} B_{\Gamma i}^{(s)} & 0 \end{pmatrix} \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} A_{i\Gamma}^{(s)} \\ B_{i\Gamma}^{(s)} \end{pmatrix}, \\
M_{pp}^{(s)} &= \begin{pmatrix} B_{\Gamma i}^{(s)} & 0 \end{pmatrix} \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} {B_{\Gamma i}^{(s)}}^T \\ 0 \end{pmatrix}, \\
\tilde{F}_{\Gamma}^{(s)} &= F_{\Gamma}^{(s)} - \begin{pmatrix} A_{\Gamma i}^{(s)} & {B_{i\Gamma}^{(s)}}^T \end{pmatrix} \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_i^{(s)} \\ 0 \end{pmatrix}, \\
\tilde{F}_i^{(s)} &= \begin{pmatrix} B_{\Gamma i}^{(s)} & 0 \end{pmatrix} \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} F_i^{(s)} \\ 0 \end{pmatrix}, \quad s = 1, \cdots, N.
\end{aligned}
$$

Lemma 1. *The subdomain Schur complements $M_{pp}^{(s)}$ and $M_{uu}^{(s)}$ are symmetric, positive semi-definite.*

Proof. Matrices $M_{pp}^{(s)}$ are clearly symmetric. Systems (2) are well-posed algebraic problems although they are not the usual Stokes because of the Dirichlet boundary condition on the pressure [2]. Therefore, for any given $P_{\Gamma}^{(s)}$, there exists

$$
\begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \end{pmatrix} = - \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T \\ B_{ii}^{(s)} & 0 \end{pmatrix}^{-1} \begin{pmatrix} {B_{i\Gamma}^{(s)}}^T \\ 0 \end{pmatrix} P_{\Gamma}^{(s)}.
$$

By Gaussian elimination, we have

$$
\begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T & {B_{\Gamma i}^{(s)}}^T \\ B_{ii}^{(s)} & 0 & 0 \\ B_{\Gamma i}^{(s)} & 0 & 0 \end{pmatrix} \begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \\ P_{\Gamma}^{(s)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -M_{pp}^{(s)} P_{\Gamma}^{(s)} \end{pmatrix}. \tag{5}
$$

Therefore,

$$
\begin{aligned}
-{P_{\Gamma}^{(s)}}^T M_{pp}^{(s)} P_{\Gamma}^{(s)} &= \begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \\ P_{\Gamma}^{(s)} \end{pmatrix}^T \begin{pmatrix} A_{ii}^{(s)} & {B_{ii}^{(s)}}^T & {B_{\Gamma i}^{(s)}}^T \\ B_{ii}^{(s)} & 0 & 0 \\ B_{\Gamma i}^{(s)} & 0 & 0 \end{pmatrix} \begin{pmatrix} U_i^{(s)} \\ P_i^{(s)} \\ P_{\Gamma}^{(s)} \end{pmatrix} \\
&= {U_i^{(s)}}^T A_{ii}^{(s)} U_i^{(s)} + 2{P_i^{(s)}}^T B_{ii}^{(s)} U_i^{(s)} + 2{P_{\Gamma}^{(s)}}^T B_{\Gamma i}^{(s)} U_i^{(s)}.
\end{aligned} \tag{6}
$$

From (5), we have

$$B_{ii}^{(s)} U_i^{(s)} = 0 \quad \text{and} \quad B_{\Gamma i}^{(s)} U_i^{(s)} = -M_{pp}^{(s)} P_\Gamma^{(s)}.$$

Then from (6) and the positivity of the matrix arising from the discretization of the Laplace operator by finite elements, we have

$$P_\Gamma^{(s)^T} M_{pp}^{(s)} P_\Gamma^{(s)} = U_i^{(s)^T} A_{ii}^{(s)} U_i^{(s)} \geq 0.$$

We also have

$$\begin{pmatrix} A_{ii}^{(s)} & B_{ii}^{(s)^T} & B_{\Gamma i}^{(s)^T} \\ B_{ii}^{(s)} & 0 & 0 \\ B_{\Gamma i}^{(s)} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1_i^{(s)} \\ 1_\Gamma^{(s)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{7}$$

where $1_i^{(s)}$ and $1_\Gamma^{(s)}$ are constants in the subdomain $\Omega^{(s)}$ and on its boundary respectively. By equality (7) one can show that in general there exists $R_p^{(s)}$ such that $M_{pp}^{(s)} R_p^{(s)} = 0$.

It is well-known that the subdomain Schur complements $M_{uu}^{(s)}$ are symmetric, positive semi-definite in general [7].

Eliminating the internal degrees of freedom from local systems (2), the interface systems (3) and the incompressibility conditions (4) can be written as

$$\begin{pmatrix} M_{uu}^{(1)} & M_{up}^{(1)} & 0 & 0 & \cdots\cdots & 0 & 0 \\ M_{pu}^{(1)} & -M_{pp}^{(1)} & 0 & 0 & \cdots\cdots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots\ \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots\ \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots\ \ddots & 0 & 0 \\ 0 & 0 & \cdots\cdots & 0 & 0 & M_{uu}^{(N)} & M_{up}^{(N)} \\ 0 & 0 & \cdots\cdots & 0 & 0 & M_{pu}^{(N)} & -M_{pp}^{(N)} \end{pmatrix} \begin{pmatrix} U_\Gamma^{(1)} \\ P_\Gamma^{(1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ U_\Gamma^{(N)} \\ P_\Gamma^{(N)} \end{pmatrix} = \begin{pmatrix} \tilde{F}_\Gamma^{(1)} \\ -\tilde{F}_i^{(1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \tilde{F}_\Gamma^{(N)} \\ -\tilde{F}_i^{(N)} \end{pmatrix}. \tag{8}$$

We introduce a unique Lagrange multiplier λ to ensure the weak continuity of the velocity on the interface as in FETI transforming the system (8) into

$$\begin{pmatrix} M_{uu}^{(1)} & M_{up}^{(1)} & 0 & 0 & \cdots & \cdots & 0 & 0 & {T^{(1)}}^T \\ M_{pu}^{(1)} & -M_{pp}^{(1)} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 & 0 & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 & M_{uu}^{(N)} & M_{up}^{(N)} & {T^{(N)}}^T \\ 0 & 0 & \cdots & \cdots & 0 & 0 & M_{pu}^{(N)} & -M_{pp}^{(N)} & 0 \\ T^{(1)} & 0 & \cdots & \cdots & \cdots & \cdots & T^{(N)} & 0 & 0 \end{pmatrix} \begin{pmatrix} U_\Gamma^{(1)} \\ P_\Gamma^{(1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ U_\Gamma^{(N)} \\ P_\Gamma^{(N)} \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{F}_\Gamma^{(1)} \\ -\tilde{F}_i^{(1)} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \tilde{F}_\Gamma^{(N)} \\ -\tilde{F}_i^{(N)} \\ 0 \end{pmatrix}. \quad (9)$$

where $\{T^{(s)}\}_{s=1,N}$ are boolean matrices of elements -1, 0 and 1. The application of the matrix $T^{(s)}$ to a matrix or a vector extracts and signs the interface components of that matrix or vector [4]. We next introduce the 0–1 matrix ${L^{(s)}}^T$ that maps the interface degrees of freedom of subdomain $\Omega^{(s)}$ into global interface degrees of freedom belonging to the interface Γ_I [9]. we develop the system (9) imposing a unique pressure on the interface as in BDD as $P_\Gamma^{(s)} = P_\Gamma$ to obtain :

$$M_{uu}^{(s)} U_\Gamma^{(s)} + M_{up}^{(s)} P_\Gamma + {T^{(s)}}^T \lambda = \tilde{F}_\Gamma^{(s)}, \quad (10)$$

$$M_{pu}^{(s)} U_\Gamma^{(s)} - M_{pp}^{(s)} P_\Gamma = -\tilde{F}^{(s)}, \quad (11)$$

$$\sum_{s=1}^{N} T^{(s)} U_\Gamma^{(s)} = 0. \quad (12)$$

We can then eliminate the degrees of freedom associated to the velocity in the equation (10) as in FETI taking into account the possibly singularity of the matrices $M_{uu}^{(s)}, s = 1, \cdots, N$. Using the previously obtained velocity into the equations (11) and (12) we get the FETI type interface system:

$$\begin{pmatrix} F_{DP} & -G_I \\ -G_I^T & 0 \end{pmatrix} \begin{pmatrix} \Lambda \\ \alpha \end{pmatrix} = \begin{pmatrix} d \\ -e^T \end{pmatrix} \quad (13)$$

where

$$F_{DP}^{(s)} = \begin{pmatrix} \left({M_{uu}^{(s)}}^+\right) & \left({M_{uu}^{(s)}}^+\right) M_{up}^{(s)} \\ M_{pu}^{(s)} \left({M_{uu}^{(s)}}^+\right) & \left(M_{pp}^{(s)} + M_{pu}^{(s)} \left({M_{uu}^{(s)}}^+\right) M_{up}^{(s)}\right) \end{pmatrix}, \quad F_{DP} = \sum_{s=1}^{N} B^{(s)} F_{DP}^{(s)} {B^{(s)}}^T,$$

$$B^{(s)} = \begin{pmatrix} T^{(s)} & 0 \\ 0 & {L^{(s)}}^T \end{pmatrix}, G_I = \begin{pmatrix} T^{(1)} R_u^{(1)} & \cdots & T^{(N_f)} R_u^{(N_f)} \\ 0 & \cdots & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda \\ P_\Gamma \end{pmatrix},$$

N_f the number of floating subdomains, $R_u^{(s)}, s = 1, \cdots, N_f$ store the basis of the kernel of the matrices $M_{uu}^{(s)}$ and α a combination of them. The interface system (13)

derives from a substructuring strategy using one-level FETI on the velocity and the primal Schur complement method on the pressure and shares some common ideas with previous methods. Indeed, the idea of combining dual and primal Schur complement method to solve algebraic systems has been introduced in [3]. A generalization of FETI and primal Schur complement has been obtained using A-FETI, a three-field variant of FETI [6]. In [8], the authors use FETI-DP on the velocity and the primal Schur complement on the pressure to solve the algebraic system arising from the discretization of Stokes with a modified Hood-Taylor element.
Interchanging the role of $U_\Gamma^{(s)}$ and $P_\Gamma^{(s)}$ we obtain the matrix

$$F_{PD}^{(s)} = \begin{pmatrix} \left(M_{uu}^{(s)} + M_{up}^{(s)} \left(M_{pp}^{(s)^+}\right) M_{pu}^{(s)}\right) & M_{up}^{(s)} \left(M_{pp}^{(s)^+}\right) \\ \left(M_{pp}^{(s)^+}\right) M_{pu}^{(s)} & \left(M_{pp}^{(s)^+}\right) \end{pmatrix}, \quad s = 1, \cdots, N. \tag{14}$$

We have

Lemma 2. *Matrices $F_{DP}^{(s)}, s = 1, \cdots, N$ are symmetric positive semi-definite.*

Proof. Matrices $F_{DP}^{(s)}, s = 1, \cdots, N$ are clearly symmetric. For any $\begin{pmatrix} \lambda^{(s)} \\ P_\Gamma^{(s)} \end{pmatrix}$ let us compute the following quantity

$$\begin{pmatrix} \lambda^{(s)} \\ P_\Gamma^{(s)} \end{pmatrix}^T F_{DP}^{(s)} \begin{pmatrix} \lambda^{(s)} \\ P_\Gamma^{(s)} \end{pmatrix} =$$

$$\begin{pmatrix} \lambda^{(s)} \\ P_\Gamma^{(s)} \end{pmatrix}^T \begin{pmatrix} T^{(s)} \left(M_{uu}^{(s)^+}\right) T^{(s)^T} & T^{(s)} \left(M_{uu}^{(s)^+}\right) M_{up}^{(s)} \\ M_{pu}^{(s)} \left(M_{uu}^{(s)^+}\right) T^{(s)^T} & \left(M_{pp}^{(s)} + M_{pu}^{(s)} \left(M_{uu}^{(s)^+}\right) M_{up}^{(s)}\right) \end{pmatrix} \begin{pmatrix} \lambda^{(s)} \\ P_\Gamma^{(s)} \end{pmatrix} =$$

$$\left\{\lambda^{(s)} + M_{up}^{(s)} P_\Gamma^{(s)}\right\}^T M_{uu}^{(s)^+} \left\{\lambda^{(s)} + M_{up}^{(s)} P_\Gamma^{(s)}\right\} + P_\Gamma^{(s)^T} M_{pp}^{(s)} P_\Gamma^{(s)}. \tag{15}$$

We have shown that matrices $M_{pp}^{(s)}$ are positive semi-definite and matrices $M_{uu}^{(s)^+}$ are known to be positive semi-definite [4]. We can then conclude by (15) that matrices $F_{DP}^{(s)}, s = 1, \cdots, N$ are positive semi-definite in general.

The FETI type operator F_{DP} is thus positive semi-definite in general and we can solve the system (13) by projected preconditioned conjugate gradient [4]. The suitable projector P is a matrix that projects Λ onto the null space of G_I^T. The preconditioner we choose is BDD with a local component defined as a weighted sum of matrices $F_{PD}^{(s)}$ and a coarse problem using the possibly kernel $\begin{pmatrix} -M_{up}^{(s)} R_p^{(s)} \\ R_p^{(s)} \end{pmatrix}$ of matrices $F_{DP}^{(s)}$. Define weights $\left\{D_u^{(s)}\right\}_{s=1,N}$ and $\left\{D_p^{(s)}\right\}_{s=1,N}$ associated with velocity and pressure respectively and the matrices

$$C = \begin{pmatrix} -D_u^{(1)} M_{up}^{(1)} R_p^{(1)} \cdots - D_u^{(N)} M_{up}^{(N)} R_p^{(N)} \\ D_p^{(1)} R_p^{(1)} \cdots D_p^{(N)} R_p^{(N)} \end{pmatrix},$$

$$B_D^{(s)} = \begin{pmatrix} D_u^{(s)} T^{(s)} & 0 \\ 0 & {L^{(s)}}^T D_p^{(s)} \end{pmatrix}, \quad s = 1, \cdots, N.$$

The BDD algorithm is defined as follows:

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1. Balance the original residual $\begin{pmatrix} r_u \\ r_p \end{pmatrix}$ by solving the auxiliary problem

$$C^T P^T F_{DP} P C \mu = C^T \begin{pmatrix} r_u \\ r_p \end{pmatrix}, \tag{16}$$

2. Compute the matrix-vector product

$$\begin{pmatrix} \bar{\lambda}^{(s)} \\ \bar{P}_\Gamma^{(s)} \end{pmatrix} = F_{PD}^{(s)} {B_D^{(s)}}^T \left(\begin{pmatrix} r_u \\ r_p \end{pmatrix} - P^T F_{DP} C \mu \right), \quad s = 1, \cdots, N, \tag{17}$$

3. Balance the residual by solving the coarse problem

$$C^T F_{DP} C \gamma = C^T \left(\begin{pmatrix} r_u \\ r_p \end{pmatrix} - P^T F_{DP} P \sum_{s=1}^{N} B_D^{(s)} \begin{pmatrix} \bar{\lambda}^{(s)} \\ \bar{P}_\Gamma^{(s)} \end{pmatrix} \right), \tag{18}$$

4. Average the solutions on the interface

$$M \begin{pmatrix} r_u \\ r_p \end{pmatrix} = \sum_{s=1}^{N} B_D^{(s)} \begin{pmatrix} \bar{\lambda}^{(s)} \\ \bar{P}_\Gamma^{(s)} \end{pmatrix} + C\gamma. \tag{19}$$

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3 Theoretical analysis of the condition number

Define $T = \sum_{s=1}^{N} B_D^{(s)} F_{PD}^{(s)} {B_D^{(s)}}^T$ and P_0 the $P^T F_{DP} P-$ orthogonal projection on the kernel of $F_{DP}^{(s)}$. Following [9] one can prove

Lemma 3. *The algorithm above returns* $z = M \begin{pmatrix} r_u \\ r_p \end{pmatrix}$*, where*

$$M = \left((Id - P_0)\, T \left(P^T F_{DP} P \right) (Id - P_0) + P_0 \right) \left(P^T F_{DP} P \right)^{-1}. \tag{20}$$

We have

Theorem 1. *The algorithm above returns* $z = M \begin{pmatrix} r_u \\ r_p \end{pmatrix}$*, where* M *is a symmetric positive definite matrix and* cond $\left(M, P^T F_{DP} P \right) \leq c$*, where*

$$c = \sup \left\{ \frac{\sum_{s=1}^{N} \left\| B^{(s)^T} P \sum_{r=1}^{N} B_D^{(r)} \begin{pmatrix} \hat{\lambda}^{(r)} \\ \hat{P}_\Gamma^{(r)} \end{pmatrix} \right\|_{F_{DP}^{(s)}}^2}{\sum_{s=1}^{N} \left\| \begin{pmatrix} \hat{\lambda}^{(s)} \\ \hat{P}_\Gamma^{(s)} \end{pmatrix} \right\|_{F_{DP}^{(s)}}^2} : G_I^T \begin{pmatrix} \hat{\lambda}^{(s)} \\ \hat{P}_\Gamma^{(s)} \end{pmatrix} = 0, \right.$$
$$\left. \left\langle \begin{pmatrix} \hat{\lambda}^{(s)} \\ \hat{P}_\Gamma^{(s)} \end{pmatrix}, \begin{pmatrix} \hat{\mu}^{(s)} \\ \hat{Q}_\Gamma^{(s)} \end{pmatrix} \right\rangle = 0, \forall \begin{pmatrix} \hat{\mu}^{(s)} \\ \hat{Q}_\Gamma^{(s)} \end{pmatrix} \in Ker(F_{DP}^{(s)}), \quad 1 \leq s \leq N \right\}. \quad (21)$$

We omit the proof of the theorem above because it essentially follows [9].

4 Conclusion

We have combined FETI and BDD to solve the discrete Stokes with continuous pressure. The original system is reduced to an interface system whose matrix is symmetric positive semi-definite in general and whose unknowns are physically homogeneous. We have given the operator form of the preconditioner and a result from which a bound for the condition number could be derived.

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